Many mathematical models involve $m + 1$ unknown functions governed by a system of $m + 1$ differential equations. In abstract form we may denote the unknowns by $u^{(0)}, \ldots, u^{(m)}$ and write the governing equations as
\[ L_0(u^{(0)}, \ldots, u^{(m)}) = 0, \]
\[
\vdots
\]
\[ L_m(u^{(0)}, \ldots, u^{(m)}) = 0, \]

where \( L_i \) is some differential operator defining differential equation number \( i \).

1 Variational forms

There are basically two ways of formulating a variational form for a system of differential equations. The first method treats each equation independently as a scalar equation, while the other method views the total system as a vector equation with a vector function as unknown.

1.1 Sequence of scalar PDEs formulation

Let us start with the approach that treats one equation at a time. We multiply equation number \( i \) by some test function \( v^{(i)} \in V^{(i)} \) and integrate over the domain:

\[
\int_{\Omega} L^{(0)}(u^{(0)}, \ldots, u^{(m)})u^{(0)} \, dx = 0, \quad (1)
\]

\[
\vdots \quad (2)
\]

\[
\int_{\Omega} L^{(m)}(u^{(0)}, \ldots, u^{(m)})u^{(m)} \, dx = 0. \quad (3)
\]

Terms with second-order derivatives may be integrated by parts, with Neumann conditions inserted in boundary integrals. Let

\[ V^{(i)} = \text{span}\{\psi_0^{(i)}, \ldots, \psi_{N_i}^{(i)}\}, \]

such that

\[ u^{(i)} = B^{(i)}(x) + \sum_{j=0}^{N_i} c_j^{(i)} \psi_j^{(i)}(x), \]

where \( B^{(i)} \) is a boundary function to handle nonzero Dirichlet conditions. Observe that different unknowns live in different spaces with different basis functions and numbers of degrees of freedom.

From the \( m \) equations in the variational forms we can derive \( m \) coupled systems of algebraic equations for the \( \Pi_{i=0}^{m} N_i \) unknown coefficients \( c_j^{(i)}, j = 0, \ldots, N_i, i = 0, \ldots, m \).
1.2 Vector PDE formulation

The alternative method for deriving a variational form for a system of differential equations introduces a vector of unknown functions

\[ u = (u^{(0)}, \ldots, u^{(m)}) \]

a vector of test functions

\[ v = (u^{(0)}, \ldots, u^{(m)}) \]

with

\[ u, v \in V = V^{(0)} \times \cdots \times V^{(m)}. \]

With nonzero Dirichlet conditions, we have a vector \( B = (B^{(0)}, \ldots, B^{(m)}) \) with boundary functions and then it is \( u - B \) that lies in \( V \), not \( u \) itself.

The governing system of differential equations is written

\[ L(u) = 0, \]

where

\[ L(u) = (L^{(0)}(u), \ldots, L^{(m)}(u)). \]

The variational form is derived by taking the inner product of the vector of equations and the test function vector:

\[ \int_{\Omega} L(u) \cdot v = 0 \quad \forall v \in V. \quad (4) \]

Observe that (4) is one scalar equation. To derive systems of algebraic equations for the unknown coefficients in the expansions of the unknown functions, one chooses \( m \) linearly independent \( v \) vectors to generate \( m \) independent variational forms from (4). The particular choice \( v = (v^{(0)}, 0, \ldots, 0) \) recovers (1), \( v = (0, \ldots, 0, v^{(m)}) \) recovers (3), and \( v = (0, \ldots, 0, v^{(i)}, 0, \ldots, 0) \) recovers the variational form number \( i \), \( \int_{\Omega} L^{(i)} v^{(i)} \, dx = 0 \), in (1)-(3).

2 A worked example

We now consider a specific system of two partial differential equations in two space dimensions:

\[ \mu \nabla^2 w = -\beta, \quad (5) \]
\[ \kappa \nabla^2 T = -\mu ||\nabla w||^2. \quad (6) \]

The unknown functions \( w(x, y) \) and \( T(x, y) \) are defined in a domain \( \Omega \), while \( \mu, \beta, \) and \( \kappa \) are given constants. The norm in (6) is the standard Euclidean norm:
\[ ||\nabla w||^2 = \nabla w \cdot \nabla w = w_x^2 + w_y^2. \]

The boundary conditions associated with (5)-(6) are \( w = 0 \) on \( \partial \Omega \) and \( T = T_0 \) on \( \partial \Omega \). Each of the equations (5) and (6) needs one condition at each point on the boundary.

The system (5)-(6) arises from fluid flow in a straight pipe, with the \( z \) axis in the direction of the pipe. The domain \( \Omega \) is a cross section of the pipe, \( w \) is the velocity in the \( z \) direction, \( \mu \) is the viscosity of the fluid, \( \beta \) is the pressure gradient along the pipe, \( T \) is the temperature, and \( \kappa \) is the heat conduction coefficient of the fluid. The equation (5) comes from the Navier-Stokes equations, and (6) follows from the energy equation. The term \( -\mu ||\nabla w||^2 \) models heating of the fluid due to internal friction.

Observe that the system (5)-(6) has only a one-way coupling: \( T \) depends on \( w \), but \( w \) does not depend on \( T \), because we can solve (5) with respect to \( w \) and then (6) with respect to \( T \). Some may argue that this is not a real system of PDEs, but just two scalar PDEs. Nevertheless, the one-way coupling is convenient when comparing different variational forms and different implementations.

### 3 Identical function spaces for the unknowns

Let us first apply the same function space \( V \) for \( w \) and \( T \) (or more precisely, \( w \in V \) and \( T - T_0 \in V \)). With

\[ V = \text{span}\{\psi_0(x, y), \ldots, \psi_N(x, y)\}, \]

we write

\[ w = \sum_{j=0}^{N} c_j^{(w)} \psi_j, \quad T = T_0 + \sum_{j=0}^{N} c_j^{(T)} \psi_j. \tag{7} \]

Note that \( w \) and \( T \) in (5)-(6) denote the exact solution of the PDEs, while \( w \) and \( T \) \( T(7) \) are the discrete functions that approximate the exact solution. It should be clear from the context whether a symbol means the exact or approximate solution, but when we need both at the same time, we use a subscript \( e \) to denote the exact solution.

#### 3.1 Variational form of each individual PDE

Inserting the expansions (7) in the governing PDEs, results in a residual in each equation,

\[ R_w = \mu \nabla^2 w + \beta, \tag{8} \]
\[ R_T = \kappa \nabla^2 T + \mu ||\nabla w||^2. \tag{9} \]

A Galerkin method demands \( R_w \) and \( R_T \) do be orthogonal to \( V \):
\[ \int_{\Omega} R_w v \, dx = 0 \quad \forall v \in V, \]
\[ \int_{\Omega} R_T v \, dx = 0 \quad \forall v \in V. \]

Because of the Dirichlet conditions, \( v = 0 \) on \( \partial \Omega \). We integrate the Laplace terms by parts and note that the boundary terms vanish since \( v = 0 \) on \( \partial \Omega \):

\[ \int_{\Omega} \mu \nabla w \cdot \nabla v \, dx = \int_{\Omega} \beta v \, dx \quad \forall v \in V, \quad (10) \]
\[ \int_{\Omega} \kappa \nabla T \cdot \nabla v \, dx = \int_{\Omega} \mu \nabla w \cdot \nabla w v \, dx \quad \forall v \in V. \quad (11) \]

### 3.2 Compound scalar variational form

The alternative way of deriving the variational from is to introduce a test vector function \( v \in V = V \times V \) and take the inner product of \( v \) and the residuals, integrated over the domain:

\[ \int_{\Omega} (R_w, R_T) \cdot v \, dx = 0 \quad \forall v \in V. \]

With \( v = (v_0, v_1) \) we get

\[ \int_{\Omega} (R_w v_0 + R_T v_1) \, dx = 0 \quad \forall v \in V. \]

Integrating the Laplace terms by parts results in

\[ \int_{\Omega} (\mu \nabla w \cdot \nabla v_0 + \kappa \nabla T \cdot \nabla v_1) \, dx = \int_{\Omega} (\beta v_0 + \mu \nabla w \cdot \nabla w v_1) \, dx, \quad \forall v \in V. \quad (12) \]

Choosing \( v_0 = v \) and \( v_1 = 0 \) gives the variational form (10), while \( v_0 = 0 \) and \( v_1 = v \) gives (11).

With the inner product notation, \( (p, q) = \int_{\Omega} pq \, dx \), we can alternatively write (10) and (11) as

\[ (\mu \nabla w, \nabla v) = (\beta, v) \quad \forall v \in V, \]
\[ (\kappa \nabla T, \nabla v) = (\mu \nabla w \cdot \nabla w, v) \quad \forall v \in V, \]

or since \( \mu \) and \( \kappa \) are considered constant,

\[ \mu(\nabla w, \nabla v) = (\beta, v) \quad \forall v \in V, \quad (13) \]
\[ \kappa(\nabla T, \nabla v) = \mu(\nabla w \cdot \nabla w, v) \quad \forall v \in V. \quad (14) \]
3.3 Decoupled linear systems

The linear systems governing the coefficients \( c_j^{(w)} \) and \( c_j^{(T)} \), \( j = 0, \ldots, N \), are derived by inserting the expansions (7) in (10) and (11), and choosing \( v = \psi_i \) for \( i = 0, \ldots, N \). The result becomes

\[
\sum_{j=0}^{N} A_{i,j}^{(w)} c_j^{(w)} = b_i^{(w)}, \quad i = 0, \ldots, N, \quad (15)
\]

\[
\sum_{j=0}^{N} A_{i,j}^{(T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N, \quad (16)
\]

\[
A_{i,j}^{(w)} = \mu(\nabla \psi_j, \nabla \psi_i), \quad (17)
\]

\[
b_i^{(w)} = (\beta, \psi_i), \quad (18)
\]

\[
A_{i,j}^{(T)} = \kappa(\nabla \psi_j, \nabla \psi_i), \quad (19)
\]

\[
b_i^{(T)} = \mu((\sum_j c_j^{(w)} \nabla \psi_j) \cdot (\sum_k c_k^{(w)} \nabla \psi_k), \psi_i). \quad (20)
\]

It can also be instructive to write the linear systems using matrices and vectors. Define \( K \) as the matrix corresponding to the Laplace operator \( \nabla^2 \). That is, \( K_{i,j} = (\nabla \psi_j, \nabla \psi_i) \). Let us introduce the vectors

\[
b^{(w)} = (b_0^{(w)}, \ldots, b_N^{(w)}), \quad (21)
\]

\[
b^{(T)} = (b_0^{(T)}, \ldots, b_N^{(T)}), \quad (22)
\]

\[
c^{(w)} = (c_0^{(w)}, \ldots, c_N^{(w)}), \quad (23)
\]

\[
c^{(T)} = (c_0^{(T)}, \ldots, c_N^{(T)}). \quad (24)
\]

The system (15)-(16) can now be expressed in matrix-vector form as

\[
\mu K c^{(w)} = b^{(w)}, \quad (21)
\]

\[
\kappa K c^{(T)} = b^{(T)}. \quad (22)
\]

We can solve the first system for \( c^{(w)} \), and then the right-hand side \( b^{(T)} \) is known such that we can solve the second system for \( c^{(T)} \).

3.4 Coupled linear systems

Despite the fact that \( w \) can be computed first, without knowing \( T \), we shall now pretend that \( w \) and \( T \) enter a two-way coupling such that we need to derive the algebraic equations as one system for all the unknowns \( c_j^{(w)} \) and \( c_j^{(T)} \), \( j = 0, \ldots, N \). This system is nonlinear in \( c_j^{(w)} \) because of the \( \nabla w \cdot \nabla w \) product.
To remove this nonlinearity, imagine that we introduce an iteration method where we replace \( \nabla w \cdot \nabla w \) by \( \nabla w - \nabla T \cdot \nabla w \), \( w \) being the \( w \) computed in the previous iteration. Then the term \( \nabla w - \nabla T \cdot \nabla w \) is linear in \( w \) since \( w \) is known.

The total linear system becomes

\[
N \sum_{j=0}^{N} A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^{N} A_{i,j}^{(w,T)} c_j^{(T)} = b_i^{(w)}, \quad i = 0, \ldots, N, \tag{23}
\]

\[
N \sum_{j=0}^{N} A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^{N} A_{i,j}^{(T,T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N, \tag{24}
\]

\[
A_{i,j}^{(w,w)} = \mu(\nabla \psi_j, \psi_i), \tag{25}
\]

\[
A_{i,j}^{(w,T)} = 0, \tag{26}
\]

\[
b_i^{(w)} = (\beta, \psi_i), \tag{27}
\]

\[
A_{i,j}^{(w,T)} = \mu((\nabla \psi w) \cdot \nabla \psi_j, \psi_i), \tag{28}
\]

\[
A_{i,j}^{(T,T)} = \kappa(\nabla \psi_j, \psi_i), \tag{29}
\]

\[
b_i^{(T)} = 0. \tag{30}
\]

This system can alternatively be written in matrix-vector form as

\[
\mu K c^{(w)} = b^{(w)}, \tag{31}
\]

\[
L c^{(w)} + \kappa K c^{(T)} = 0, \tag{32}
\]

with \( L \) as the matrix from the \( \nabla w \cdot \nabla \) operator: \( L_{i,j} = A_{i,j}^{(w,T)} \).

The matrix-vector equations are often conveniently written in block form:

\[
\begin{pmatrix}
\mu K & 0 \\
L & \kappa K
\end{pmatrix}
\begin{pmatrix}
c^{(w)} \\
c^{(T)}
\end{pmatrix} =
\begin{pmatrix}
b^{(w)} \\
0
\end{pmatrix},
\]

Note that in the general case where all unknowns enter all equations, we have to solve the compound system \((23)-(24)\) since then we cannot utilize the special property that \((15)\) does not involve \( T \) and can be solved first.

When the viscosity depends on the temperature, the \( \mu \nabla^2 w \) term must be replaced by \( \nabla \cdot (\mu(T) \nabla w) \), and then \( T \) enters the equation for \( w \). Now we have a two-way coupling since both equations contain \( w \) and \( T \) and therefore must be solved simultaneously. The equation \( \nabla \cdot (\mu(T) \nabla w) = -\beta \) is nonlinear, and if some iteration procedure is invoked, where we use a previously computed \( T \) in the viscosity \( \mu(T_-) \), the coefficient is known, and the equation involves only one unknown, \( w \). In that case we are back to the one-way coupled set of PDEs.

We may also formulate our PDE system as a vector equation. To this end, we introduce the vector of unknowns \( u = (u^{(0)}, u^{(1)}) \), where \( u^{(0)} = w \) and \( u^{(1)} = T \). We then have
\[ \nabla^2 u = \left( -\kappa^{-1} \mu \nabla u(0) \cdot \nabla u(0) \right). \]

4 Different function spaces for the unknowns

It is easy to generalize the previous formulation to the case where \( w \in V^{(w)} \) and \( T \in V^{(T)} \), where \( V^{(w)} \) and \( V^{(T)} \) can be different spaces with different numbers of degrees of freedom. For example, we may use quadratic basis functions for \( w \) and linear for \( T \). Approximation of the unknowns by different finite element spaces is known as \textit{mixed finite element methods}.

We write

\[ V^{(w)} = \text{span}\{\psi_0^{(w)}, \ldots, \psi_{N_w}^{(w)}\}, \]
\[ V^{(T)} = \text{span}\{\psi_0^{(T)}, \ldots, \psi_{N_T}^{(T)}\}. \]

The next step is to multiply (5) by a test function \( v^{(w)} \in V^{(w)} \) and (6) by a \( v^{(T)} \in V^{(T)} \), integrate by parts and arrive at

\[ \int_{\Omega} \mu \nabla w \cdot \nabla v^{(w)} \, dx = \int_{\Omega} \beta v^{(w)} \, dx \quad \forall v^{(w)} \in V^{(w)}, \quad (33) \]
\[ \int_{\Omega} \kappa \nabla T \cdot \nabla v^{(T)} \, dx = \int_{\Omega} \mu \nabla w \cdot \nabla v^{(T)} \, dx \quad \forall v^{(T)} \in V^{(T)}. \quad (34) \]

The compound scalar variational formulation applies a test vector function \( v = (v^{(w)}, v^{(T)}) \) and reads

\[ \int_{\Omega} (\mu \nabla w \cdot \nabla v^{(w)} + \kappa \nabla T \cdot \nabla v^{(T)}) \, dx = \int_{\Omega} (\beta v^{(w)} + \mu \nabla w \cdot \nabla v^{(T)}) \, dx, \quad (35) \]

valid \( \forall v \in V = V^{(w)} \times V^{(T)} \).

The associated linear system is similar to (15)-(16) or (23)-(24), except that we need to distinguish between \( \psi_i^{(w)} \) and \( \psi_i^{(T)} \), and the range in the sums over \( j \) must match the number of degrees of freedom in the spaces \( V^{(w)} \) and \( V^{(T)} \). The formulas become
\[
\sum_{j=0}^{N_w} A_{i,j}^{(w)} c_j^{(w)} = b_i^{(w)}, \quad i = 0, \ldots, N_w, \tag{36}
\]
\[
\sum_{j=0}^{N_T} A_{i,j}^{(T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N_T, \tag{37}
\]
\[
A_{i,j}^{(w)} = \mu(\nabla \psi_j^{(w)}, \psi_i^{(w)}), \tag{38}
\]
\[
b_i^{(w)} = (\beta, \psi_i^{(w)}), \tag{39}
\]
\[
A_{i,j}^{(T)} = \kappa(\nabla \psi_j^{(T)}, \psi_i^{(T)}), \tag{40}
\]
\[
b_i^{(T)} = \mu(\nabla w_-, \psi_i^{(T)}). \tag{41}
\]

In the case we formulate one compound linear system involving both \(c_j^{(w)}, j = 0, \ldots, N_w,\) and \(c_j^{(T)}, j = 0, \ldots, N_T,\) (23)-(24) becomes

\[
\sum_{j=0}^{N_w} A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^{N_T} A_{i,j}^{(w,T)} c_j^{(T)} = b_i^{(w)}, \quad i = 0, \ldots, N_w, \tag{42}
\]
\[
\sum_{j=0}^{N_w} A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^{N_T} A_{i,j}^{(T,T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N_T, \tag{43}
\]
\[
A_{i,j}^{(w,w)} = \mu(\nabla \psi_j^{(w)}, \psi_i^{(w)}), \tag{44}
\]
\[
A_{i,j}^{(w,T)} = 0, \tag{45}
\]
\[
b_i^{(w)} = (\beta, \psi_i^{(w)}), \tag{46}
\]
\[
A_{i,j}^{(T,w)} = \mu(\nabla w_-, \nabla \psi_j^{(w)}), \tag{47}
\]
\[
A_{i,j}^{(T,T)} = \kappa(\nabla \psi_j^{(T)}, \psi_i^{(T)}), \tag{48}
\]
\[
b_i^{(T)} = 0. \tag{49}
\]

The corresponding block form

\[
\begin{pmatrix}
\mu K^{(w)} & 0 \\
L & \kappa K^{(T)}
\end{pmatrix}
\begin{pmatrix}
c^{(w)} \\
c^{(T)}
\end{pmatrix}
= \begin{pmatrix}
b^{(w)} \\
0
\end{pmatrix},
\]

has square and rectangular block matrices: \(K^{(w)}\) is \(N_w \times N_w,\) \(K^{(T)}\) is \(N_T \times N_T,\) while \(L\) is \(N_T \times N_w,\)

### 5 Computation in 1D

We can reduce the system (5)-(6) to one space dimension, which corresponds to flow in a channel between two flat plates. Alternatively, one may consider
flow in a circular pipe, introduce cylindrical coordinates, and utilize the radial symmetry to reduce the equations to a one-dimensional problem in the radial coordinate. The former model becomes

\[ \mu w_{xx} = -\beta, \]  
\[ \kappa T_{xx} = -\mu w_x^2, \]  
while the model in the radial coordinate \( r \) reads

\[ \mu \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) = -\beta, \]  
\[ \kappa \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = -\mu \left( \frac{dw}{dr} \right)^2. \]

The domain for (50)-(51) is \( \Omega = [0, H] \), with boundary conditions \( w(0) = w(H) = 0 \) and \( T(0) = T(H) = T_0 \). For (52)-(53) the domain is \( [0, R] \) (\( R \) being the radius of the pipe) and the boundary conditions are \( d\mu/dr = dT/dr = 0 \) for \( r = 0 \), \( u(R) = 0 \), and \( T(R) = T_0 \).

Calculations to be continued...

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