Variational forms for systems of PDEs

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PRELIMINARY VERSION

Contents

Many mathematical models involve $m+1$ unknown functions governed by a system of $m+1$ differential equations. In abstract form we may denote the unknowns by $u^{(0)},\ldots,u^{(m)}$ and write the governing equations as

$$
\begin{aligned}
\mathcal{L}_0(u^{(0)}, \ldots, u^{(m)}) &= 0, \\
&\vdots \\
\mathcal{L}_m(u^{(0)}, \ldots, u^{(m)}) &= 0,
\end{aligned}
$$

where $\mathcal{L}_i$ is some differential operator defining differential equation number i.

1 Variational forms

There are basically two ways of formulating a variational form for a system of differential equations. The first method treats each equation independently as a scalar equation, while the other method views the total system as a vector equation with a vector function as unknown.

1.1 Sequence of scalar PDEs formulation

Let us start with the approach that treats one equation at a time. We multiply equation number i by some test function $v^{(i)} \in V^{(i)}$ and integrate over the domain:

$$
\int_\Omega \mathcal{L}^{(0)}(u^{(0)}, \ldots, u^{(m)}) v^{(0)} \,\mathrm{d}x = 0, \tag{1}
$$

$$
\vdots \tag{2}
$$

$$
\int_\Omega \mathcal{L}^{(m)}(u^{(0)}, \ldots, u^{(m)}) v^{(m)} \,\mathrm{d}x = 0\,. \tag{3}
$$

Terms with second-order derivatives may be integrated by parts, with Neumann conditions inserted in boundary integrals. Let

$$
V^{(i)} = \mathrm{span}\{\psi_0^{(i)}, \ldots, \psi_{N_i}^{(i)}\},
$$

such that

$$
u^{(i)} = B^{(i)}(\boldsymbol{x}) + \sum_{j=0}^{N_i} c_j^{(i)} \psi_j^{(i)}(\boldsymbol{x}),
$$

where $B^{(i)}$ is a boundary function to handle nonzero Dirichlet conditions. Observe that different unknowns live in different spaces with different basis functions and numbers of degrees of freedom.

From the m equations in the variational forms we can derive m coupled systems of algebraic equations for the $\Pi_{i=0}^{m} N_i$ unknown coefficients $c_j^{(i)}$, $j = 0, \ldots, N_i$, $i = 0, \ldots, m$.

1.2 Vector PDE formulation

The alternative method for deriving a variational form for a system of differential equations introduces a vector of unknown functions

$$\boldsymbol{u} = (u^{(0)}, \ldots, u^{(m)}),$$

a vector of test functions

$$\boldsymbol{v} = (u^{(0)}, \ldots, u^{(m)}),$$

with

$$\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V} = V^{(0)} \times \cdots \times V^{(m)} .$$

With nonzero Dirichlet conditions, we have a vector $\boldsymbol{B} = (B^{(0)}, \ldots, B^{(m)})$ with boundary functions and then it is $\boldsymbol{u} - \boldsymbol{B}$ that lies in $\boldsymbol{V}$, not $\boldsymbol{u}$ itself.

The governing system of differential equations is written

$$\boldsymbol{\mathcal{L}}(\boldsymbol{u}) = 0,$$

where

$$\boldsymbol{\mathcal{L}}(\boldsymbol{u}) = (\mathcal{L}^{(0)}(\boldsymbol{u}), \ldots, \mathcal{L}^{(m)}(\boldsymbol{u})) .$$

The variational form is derived by taking the inner product of the vector of equations and the test function vector:

$$\int_\Omega \boldsymbol{\mathcal{L}}(\boldsymbol{u}) \cdot \boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V} . \tag{4}$$

Observe that (4) is one scalar equation. To derive systems of algebraic equations for the unknown coefficients in the expansions of the unknown functions, one chooses m linearly independent $\boldsymbol{v}$ vectors to generate m independent variational forms from (4). The particular choice $\boldsymbol{v} = (v^{(0)}, 0, \ldots, 0)$ recovers (1), $\boldsymbol{v} = (0, \ldots, 0, v^{(m)}$ recovers (3), and $\boldsymbol{v} = (0, \ldots, 0, v^{(i)}, 0, \ldots, 0)$ recovers the variational form number i, $\int_\Omega \mathcal{L}^{(i)} v^{(i)} \,\mathrm{d}x = 0$, in (1)-(3).

2 A worked example

We now consider a specific system of two partial differential equations in two space dimensions:

$$\mu\nabla^2 w = -\beta, \tag{5}$$
$$\kappa\nabla^2 T = -\mu||\nabla w||^2 . \tag{6}$$

The unknown functions $w(x, y)$ and $T(x, y)$ are defined in a domain Ω, while μ, β, and κ are given constants. The norm in (6) is the standard Euclidean norm:

$$||\nabla w||^2 = \nabla w \cdot \nabla w = w_x^2 + w_y^2 \, .$$

The boundary conditions associated with (5)-(6) are $w = 0$ on $\partial\Omega$ and $T = T_0$ on $\partial\Omega$. Each of the equations (5) and (6) needs one condition at each point on the boundary.

The system (5)-(6) arises from fluid flow in a straight pipe, with the z axis in the direction of the pipe. The domain Ω is a cross section of the pipe, w is the velocity in the z direction, μ is the viscosity of the fluid, β is the pressure gradient along the pipe, T is the temperature, and κ is the heat conduction coefficient of the fluid. The equation (5) comes from the Navier-Stokes equations, and (6) follows from the energy equation. The term $-\mu||\nabla w||^2$ models heating of the fluid due to internal friction.

Observe that the system (5)-(6) has only a one-way coupling: T depends on w, but w does not depend on T, because we can solve (5) with respect to w and then (6) with respect to T. Some may argue that this is not a real system of PDEs, but just two scalar PDEs. Nevertheless, the one-way coupling is convenient when comparing different variational forms and different implementations.

3 Identical function spaces for the unknowns

Let us first apply the same function space V for w and T (or more precisely, $w \in V$ and $T - T_0 \in V$). With

$$V = \mathrm{span}\{\psi_0(x,y), \ldots, \psi_N(x,y)\},$$

we write

$$w = \sum_{j=0}^{N} c_j^{(w)} \psi_j, \quad T = T_0 + \sum_{j=0}^{N} c_j^{(T)} \psi_j \, . \tag{7}$$

Note that w and T in (5)-(6) denote the exact solution of the PDEs, while w and T (7) are the discrete functions that approximate the exact solution. It should be clear from the context whether a symbol means the exact or approximate solution, but when we need both at the same time, we use a subscript e to denote the exact solution.

3.1 Variational form of each individual PDE

Inserting the expansions (7) in the governing PDEs, results in a residual in each equation,

$$R_w = \mu\nabla^2 w + \beta, \tag{8}$$

$$R_T = \kappa\nabla^2 T + \mu||\nabla w||^2 \, . \tag{9}$$

A Galerkin method demands R_w and R_T do be orthogonal to V:

$$\int_\Omega R_w v \,\mathrm{d}x = 0 \quad \forall v \in V,$$
$$\int_\Omega R_T v \,\mathrm{d}x = 0 \quad \forall v \in V\,.$$

Because of the Dirichlet conditions, $v = 0$ on $\partial\Omega$. We integrate the Laplace terms by parts and note that the boundary terms vanish since $v = 0$ on $\partial\Omega$:

$$\int_\Omega \mu \nabla w \cdot \nabla v \,\mathrm{d}x = \int_\Omega \beta v \,\mathrm{d}x \quad \forall v \in V, \tag{10}$$
$$\int_\Omega \kappa \nabla T \cdot \nabla v \,\mathrm{d}x = \int_\Omega \mu \nabla w \cdot \nabla w \, v \,\mathrm{d}x \quad \forall v \in V\,. \tag{11}$$

3.2 Compound scalar variational form

The alternative way of deriving the variational from is to introduce a test vector function $\boldsymbol{v} \in \boldsymbol{V} = V \times V$ and take the inner product of $\boldsymbol{v}$ and the residuals, integrated over the domain:

$$\int_\Omega (R_w, R_T) \cdot \boldsymbol{v} \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

With $\boldsymbol{v} = (v_0, v_1)$ we get

$$\int_\Omega (R_w v_0 + R_T v_1) \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

Integrating the Laplace terms by parts results in

$$\int_\Omega (\mu \nabla w \cdot \nabla v_0 + \kappa \nabla T \cdot \nabla v_1) \,\mathrm{d}x = \int_\Omega (\beta v_0 + \mu \nabla w \cdot \nabla w \, v_1) \,\mathrm{d}x, \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,. \tag{12}$$

Choosing $v_0 = v$ and $v_1 = 0$ gives the variational form (10), while $v_0 = 0$ and $v_1 = v$ gives (11).

With the inner product notation, $(p, q) = \int_\Omega pq \,\mathrm{d}x$, we can alternatively write (10) and (11) as

$$(\mu \nabla w, \nabla v) = (\beta, v) \quad \forall v \in V,$$
$$(\kappa \nabla T, \nabla v) = (\mu \nabla w \cdot \nabla w, v) \quad \forall v \in V,$$

or since μ and κ are considered constant,

$$\mu (\nabla w, \nabla v) = (\beta, v) \quad \forall v \in V, \tag{13}$$
$$\kappa (\nabla T, \nabla v) = \mu (\nabla w \cdot \nabla w, v) \quad \forall v \in V\,. \tag{14}$$

3.3 Decoupled linear systems

The linear systems governing the coefficients $c_j^{(w)}$ and $c_j^{(T)}$, $j = 0, \ldots, N$, are derived by inserting the expansions (7) in (10) and (11), and choosing $v = \psi_i$ for $i = 0, \ldots, N$. The result becomes

$$\sum_{j=0}^{N} A_{i,j}^{(w)} c_j^{(w)} = b_i^{(w)}, \quad i = 0, \ldots, N, \tag{15}$$
$$\sum_{j=0}^{N} A_{i,j}^{(T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N, \tag{16}$$
$$A_{i,j}^{(w)} = \mu(\nabla\psi_j, \nabla\psi_i), \tag{17}$$
$$b_i^{(w)} = (\beta, \psi_i), \tag{18}$$
$$A_{i,j}^{(T)} = \kappa(\nabla\psi_j, \nabla\psi_i), \tag{19}$$
$$b_i^{(T)} = \mu((\sum_j c_j^{(w)} \nabla\psi_j) \cdot (\sum_k c_k^{(w)} \nabla\psi_k), \psi_i)\,. \tag{20}$$

It can also be instructive to write the linear systems using matrices and vectors. Define K as the matrix corresponding to the Laplace operator ∇^2. That is, $K_{i,j} = (\nabla\psi_j, \nabla\psi_i)$. Let us introduce the vectors

$$b^{(w)} = (b_0^{(w)}, \ldots, b_N^{(w)}),$$
$$b^{(T)} = (b_0^{(T)}, \ldots, b_N^{(T)}),$$
$$c^{(w)} = (c_0^{(w)}, \ldots, c_N^{(w)}),$$
$$c^{(T)} = (c_0^{(T)}, \ldots, c_N^{(T)})\,.$$

The system (15)-(16) can now be expressed in matrix-vector form as

$$\mu K c^{(w)} = b^{(w)}, \tag{21}$$
$$\kappa K c^{(T)} = b^{(T)}\,. \tag{22}$$

We can solve the first system for $c^{(w)}$, and then the right-hand side $b^{(T)}$ is known such that we can solve the second system for $c^{(T)}$.

3.4 Coupled linear systems

Despite the fact that w can be computed first, without knowing T, we shall now pretend that w and T enter a two-way coupling such that we need to derive the algebraic equations as *one system* for all the unknowns $c_j^{(w)}$ and $c_j^{(T)}$, $j = 0, \ldots, N$. This system is nonlinear in $c_j^{(w)}$ because of the $\nabla w \cdot \nabla w$ product.

To remove this nonlinearity, imagine that we introduce an iteration method where we replace $\nabla w \cdot \nabla w$ by $\nabla w_{-} \cdot \nabla w$, w_{-} being the w computed in the previous iteration. Then the term $\nabla w_{-} \cdot \nabla w$ is linear in w since w_{-} is known. The total linear system becomes

$$
\begin{aligned}
\sum_{j=0}^{N} A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^{N} A_{i,j}^{(w,T)} c_j^{(T)} &= b_i^{(w)}, \quad i = 0, \ldots, N, && (23)\\
\sum_{j=0}^{N} A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^{N} A_{i,j}^{(T,T)} c_j^{(T)} &= b_i^{(T)}, \quad i = 0, \ldots, N, && (24)\\
A_{i,j}^{(w,w)} &= \mu(\nabla\psi_j, \psi_i), && (25)\\
A_{i,j}^{(w,T)} &= 0, && (26)\\
b_i^{(w)} &= (\beta, \psi_i), && (27)\\
A_{i,j}^{(w,T)} &= \mu((\nabla\psi w_{-}) \cdot \nabla\psi_j), \psi_i), && (28)\\
A_{i,j}^{(T,T)} &= \kappa(\nabla\psi_j, \psi_i), && (29)\\
b_i^{(T)} &= 0\,. && (30)
\end{aligned}
$$

This system can alternatively be written in matrix-vector form as

$$
\begin{aligned}
\mu K c^{(w)} &= b^{(w)}, && (31)\\
L c^{(w)} + \kappa K c^{(T)} &= 0, && (32)
\end{aligned}
$$

with L as the matrix from the $\nabla w_{-} \cdot \nabla$ operator: $L_{i,j} = A_{i,j}^{(w,T)}$.

The matrix-vector equations are often conveniently written in block form:

$$
\left(\begin{array}{cc} \mu K & 0 \\ L & \kappa K \end{array}\right) \left(\begin{array}{c} c^{(w)} \\ c^{(T)} \end{array}\right) = \left(\begin{array}{c} b^{(w)} \\ 0 \end{array}\right),
$$

Note that in the general case where all unknowns enter all equations, we have to solve the compound system (23)-(24) since then we cannot utilize the special property that (15) does not involve T and can be solved first.

When the viscosity depends on the temperature, the $\mu\nabla^2 w$ term must be replaced by $\nabla \cdot (\mu(T)\nabla w)$, and then T enters the equation for w. Now we have a two-way coupling since both equations contain w and T and therefore must be solved simultaneously Th equation $\nabla \cdot (\mu(T)\nabla w) = -\beta$ is nonlinear, and if some iteration procedure is invoked, where we use a previously computed T_{-} in the viscosity ($\mu(T_{-})$), the coefficient is known, and the equation involves only one unknown, w. In that case we are back to the one-way coupled set of PDEs.

We may also formulate our PDE system as a vector equation. To this end, we introduce the vector of unknowns $\boldsymbol{u} = (u^{(0)}, u^{(1)})$, where $u^{(0)} = w$ and $u^{(1)} = T$. We then have

$$\nabla^2 \boldsymbol{u} = \begin{pmatrix} -\mu^{-1}\beta \\ -\kappa^{-1}\mu\nabla u^{(0)} \cdot \nabla u^{(0)} \end{pmatrix} .$$

4 Different function spaces for the unknowns

It is easy to generalize the previous formulation to the case where $w \in V^{(w)}$ and $T \in V^{(T)}$, where $V^{(w)}$ and $V^{(T)}$ can be different spaces with different numbers of degrees of freedom. For example, we may use quadratic basis functions for w and linear for T. Approximation of the unknowns by different finite element spaces is known as *mixed finite element methods*.

We write

$$\begin{aligned} V^{(w)} &= \text{span}\{\psi_0^{(w)}, \ldots, \psi_{N_w}^{(w)}\}, \\ V^{(T)} &= \text{span}\{\psi_0^{(T)}, \ldots, \psi_{N_T}^{(T)}\} . \end{aligned}$$

The next step is to multiply (5) by a test function $v^{(w)} \in V^{(w)}$ and (6) by a $v^{(T)} \in V^{(T)}$, integrate by parts and arrive at

$$\int_\Omega \mu\nabla w \cdot \nabla v^{(w)} \,\mathrm{d}x = \int_\Omega \beta v^{(w)} \,\mathrm{d}x \quad \forall v^{(w)} \in V^{(w)}, \tag{33}$$

$$\int_\Omega \kappa\nabla T \cdot \nabla v^{(T)} \,\mathrm{d}x = \int_\Omega \mu\nabla w \cdot \nabla w \, v^{(T)} \,\mathrm{d}x \quad \forall v^{(T)} \in V^{(T)} . \tag{34}$$

The compound scalar variational formulation applies a test vector function $\boldsymbol{v} = (v^{(w)}, v^{(T)})$ and reads

$$\int_\Omega (\mu\nabla w \cdot \nabla v^{(w)} + \kappa\nabla T \cdot \nabla v^{(T)}) \,\mathrm{d}x = \int_\Omega (\beta v^{(w)} + \mu\nabla w \cdot \nabla w \, v^{(T)}) \,\mathrm{d}x, \tag{35}$$

valid $\forall \boldsymbol{v} \in \boldsymbol{V} = V^{(w)} \times V^{(T)}$.

The associated linear system is similar to (15)-(16) or (23)-(24), except that we need to distinguish between $\psi_i^{(w)}$ and $\psi_i^{(T)}$, and the range in the sums over j must match the number of degrees of freedom in the spaces $V^{(w)}$ and $V^{(T)}$. The formulas become

$$\sum_{j=0}^{N_w} A_{i,j}^{(w)} c_j^{(w)} = b_i^{(w)}, \quad i = 0, \ldots, N_w, \tag{36}$$

$$\sum_{j=0}^{N_T} A_{i,j}^{(T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N_T, \tag{37}$$

$$A_{i,j}^{(w)} = \mu(\nabla \psi_j^{(w)}, \psi_i^{(w)}), \tag{38}$$

$$b_i^{(w)} = (\beta, \psi_i^{(w)}), \tag{39}$$

$$A_{i,j}^{(T)} = \kappa(\nabla \psi_j^{(T)}, \psi_i^{(T)}), \tag{40}$$

$$b_i^{(T)} = \mu(\nabla w_-, \psi_i^{(T)})\,. \tag{41}$$

In the case we formulate one compound linear system involving both $c_j^{(w)}$, $j = 0, \ldots, N_w$, and $c_j^{(T)}$, $j = 0, \ldots, N_T$, (23)-(24) becomes

$$\sum_{j=0}^{N_w} A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^{N_T} A_{i,j}^{(w,T)} c_j^{(T)} = b_i^{(w)}, \quad i = 0, \ldots, N_w, \tag{42}$$

$$\sum_{j=0}^{N_w} A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^{N_T} A_{i,j}^{(T,T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \ldots, N_T, \tag{43}$$

$$A_{i,j}^{(w,w)} = \mu(\nabla \psi_j^{(w)}, \psi_i^{(w)}), \tag{44}$$

$$A_{i,j}^{(w,T)} = 0, \tag{45}$$

$$b_i^{(w)} = (\beta, \psi_i^{(w)}), \tag{46}$$

$$A_{i,j}^{(w,T)} = \mu(\nabla w_- \cdot \nabla \psi_j^{(w)}), \psi_i^{(T)}), \tag{47}$$

$$A_{i,j}^{(T,T)} = \kappa(\nabla \psi_j^{(T)}, \psi_i^{(T)}), \tag{48}$$

$$b_i^{(T)} = 0\,. \tag{49}$$

The corresponding block form

$$\begin{pmatrix} \mu K^{(w)} & 0 \\ L & \kappa K^{(T)} \end{pmatrix} \begin{pmatrix} c^{(w)} \\ c^{(T)} \end{pmatrix} = \begin{pmatrix} b^{(w)} \\ 0 \end{pmatrix},$$

has square and rectangular block matrices: $K^{(w)}$ is $N_w \times N_w$, $K^{(T)}$ is $N_T \times N_T$, while L is $N_T \times N_w$,

5 Computations in 1D

We can reduce the system (5)-(6) to one space dimension, which corresponds to flow in a channel between two flat plates. Alternatively, one may consider flow in a circular pipe, introduce cylindrical coordinates, and utilize the radial symmetry to reduce the equations to a one-dimensional problem in the radial coordinate. The former model becomes

$$\mu w_{xx} = -\beta, \tag{50}$$

$$\kappa T_{xx} = -\mu w_x^2, \tag{51}$$

while the model in the radial coordinate r reads

$$\mu \frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) = -\beta, \tag{52}$$

$$\kappa \frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = -\mu \left(\frac{dw}{dr} \right)^2 . \tag{53}$$

The domain for (50)-(51) is $\Omega = [0, H]$, with boundary conditions $w(0) = w(H) = 0$ and $T(0) = T(H) = T_0$. For (52)-(53) the domain is $[0, R]$ (R being the radius of the pipe) and the boundary conditions are $du/dr = dT/dr = 0$ for $r = 0$, $u(R) = 0$, and $T(R) = T_0$.

Calculations to be continued...

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