

# Variational forms for systems of PDEs

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Many mathematical models involve  $m + 1$  unknown functions governed by a system of  $m + 1$  differential equations. In abstract form we may denote the unknowns by  $u^{(0)}, \dots, u^{(m)}$  and write the governing equations as

$$\begin{aligned}\mathcal{L}_0(u^{(0)}, \dots, u^{(m)}) &= 0, \\ &\vdots \\ \mathcal{L}_m(u^{(0)}, \dots, u^{(m)}) &= 0,\end{aligned}$$

where  $\mathcal{L}_i$  is some differential operator defining differential equation number  $i$ .

## 1 Variational forms

There are basically two ways of formulating a variational form for a system of differential equations. The first method treats each equation independently as a scalar equation, while the other method views the total system as a vector equation with a vector function as unknown.

### 1.1 Sequence of scalar PDEs formulation

Let us start with the approach that treats one equation at a time. We multiply equation number  $i$  by some test function  $v^{(i)} \in V^{(i)}$  and integrate over the domain:

$$\int_{\Omega} \mathcal{L}^{(0)}(u^{(0)}, \dots, u^{(m)})v^{(0)} \, dx = 0, \tag{1}$$

$$\vdots \tag{2}$$

$$\int_{\Omega} \mathcal{L}^{(m)}(u^{(0)}, \dots, u^{(m)})v^{(m)} \, dx = 0. \tag{3}$$

Terms with second-order derivatives may be integrated by parts, with Neumann conditions inserted in boundary integrals. Let

$$V^{(i)} = \text{span}\{\psi_0^{(i)}, \dots, \psi_{N_i}^{(i)}\},$$

such that

$$u^{(i)} = B^{(i)}(\mathbf{x}) + \sum_{j=0}^{N_i} c_j^{(i)} \psi_j^{(i)}(\mathbf{x}),$$

where  $B^{(i)}$  is a boundary function to handle nonzero Dirichlet conditions. Observe that different unknowns live in different spaces with different basis functions and numbers of degrees of freedom.

From the  $m$  equations in the variational forms we can derive  $m$  coupled systems of algebraic equations for the  $\prod_{i=0}^m N_i$  unknown coefficients  $c_j^{(i)}$ ,  $j = 0, \dots, N_i$ ,  $i = 0, \dots, m$ .

## 1.2 Vector PDE formulation

The alternative method for deriving a variational form for a system of differential equations introduces a vector of unknown functions

$$\mathbf{u} = (u^{(0)}, \dots, u^{(m)}),$$

a vector of test functions

$$\mathbf{v} = (v^{(0)}, \dots, v^{(m)}),$$

with

$$\mathbf{u}, \mathbf{v} \in \mathbf{V} = V^{(0)} \times \dots \times V^{(m)}.$$

With nonzero Dirichlet conditions, we have a vector  $\mathbf{B} = (B^{(0)}, \dots, B^{(m)})$  with boundary functions and then it is  $\mathbf{u} - \mathbf{B}$  that lies in  $\mathbf{V}$ , not  $\mathbf{u}$  itself.

The governing system of differential equations is written

$$\mathcal{L}(\mathbf{u}) = 0,$$

where

$$\mathcal{L}(\mathbf{u}) = (\mathcal{L}^{(0)}(\mathbf{u}), \dots, \mathcal{L}^{(m)}(\mathbf{u})).$$

The variational form is derived by taking the inner product of the vector of equations and the test function vector:

$$\int_{\Omega} \mathcal{L}(\mathbf{u}) \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in \mathbf{V}. \quad (4)$$

Observe that (4) is one scalar equation. To derive systems of algebraic equations for the unknown coefficients in the expansions of the unknown functions, one chooses  $m$  linearly independent  $\mathbf{v}$  vectors to generate  $m$  independent variational forms from (4). The particular choice  $\mathbf{v} = (v^{(0)}, 0, \dots, 0)$  recovers (1),  $\mathbf{v} = (0, \dots, 0, v^{(m)})$  recovers (3), and  $\mathbf{v} = (0, \dots, 0, v^{(i)}, 0, \dots, 0)$  recovers the variational form number  $i$ ,  $\int_{\Omega} \mathcal{L}^{(i)} v^{(i)} dx = 0$ , in (1)-(3).

## 2 A worked example

We now consider a specific system of two partial differential equations in two space dimensions:

$$\mu \nabla^2 w = -\beta, \quad (5)$$

$$\kappa \nabla^2 T = -\mu \|\nabla w\|^2. \quad (6)$$

The unknown functions  $w(x, y)$  and  $T(x, y)$  are defined in a domain  $\Omega$ , while  $\mu$ ,  $\beta$ , and  $\kappa$  are given constants. The norm in (6) is the standard Euclidean norm:

$$\|\nabla w\|^2 = \nabla w \cdot \nabla w = w_x^2 + w_y^2.$$

The boundary conditions associated with (5)-(6) are  $w = 0$  on  $\partial\Omega$  and  $T = T_0$  on  $\partial\Omega$ . Each of the equations (5) and (6) needs one condition at each point on the boundary.

The system (5)-(6) arises from fluid flow in a straight pipe, with the  $z$  axis in the direction of the pipe. The domain  $\Omega$  is a cross section of the pipe,  $w$  is the velocity in the  $z$  direction,  $\mu$  is the viscosity of the fluid,  $\beta$  is the pressure gradient along the pipe,  $T$  is the temperature, and  $\kappa$  is the heat conduction coefficient of the fluid. The equation (5) comes from the Navier-Stokes equations, and (6) follows from the energy equation. The term  $-\mu\|\nabla w\|^2$  models heating of the fluid due to internal friction.

Observe that the system (5)-(6) has only a one-way coupling:  $T$  depends on  $w$ , but  $w$  does not depend on  $T$ , because we can solve (5) with respect to  $w$  and then (6) with respect to  $T$ . Some may argue that this is not a real system of PDEs, but just two scalar PDEs. Nevertheless, the one-way coupling is convenient when comparing different variational forms and different implementations.

### 3 Identical function spaces for the unknowns

Let us first apply the same function space  $V$  for  $w$  and  $T$  (or more precisely,  $w \in V$  and  $T - T_0 \in V$ ). With

$$V = \text{span}\{\psi_0(x, y), \dots, \psi_N(x, y)\},$$

we write

$$w = \sum_{j=0}^N c_j^{(w)} \psi_j, \quad T = T_0 + \sum_{j=0}^N c_j^{(T)} \psi_j. \quad (7)$$

Note that  $w$  and  $T$  in (5)-(6) denote the exact solution of the PDEs, while  $w$  and  $T$  (7) are the discrete functions that approximate the exact solution. It should be clear from the context whether a symbol means the exact or approximate solution, but when we need both at the same time, we use a subscript e to denote the exact solution.

#### 3.1 Variational form of each individual PDE

Inserting the expansions (7) in the governing PDEs, results in a residual in each equation,

$$R_w = \mu \nabla^2 w + \beta, \quad (8)$$

$$R_T = \kappa \nabla^2 T + \mu \|\nabla w\|^2. \quad (9)$$

A Galerkin method demands  $R_w$  and  $R_T$  do be orthogonal to  $V$ :

$$\begin{aligned}\int_{\Omega} R_w v \, dx &= 0 \quad \forall v \in V, \\ \int_{\Omega} R_T v \, dx &= 0 \quad \forall v \in V.\end{aligned}$$

Because of the Dirichlet conditions,  $v = 0$  on  $\partial\Omega$ . We integrate the Laplace terms by parts and note that the boundary terms vanish since  $v = 0$  on  $\partial\Omega$ :

$$\int_{\Omega} \mu \nabla w \cdot \nabla v \, dx = \int_{\Omega} \beta v \, dx \quad \forall v \in V, \quad (10)$$

$$\int_{\Omega} \kappa \nabla T \cdot \nabla v \, dx = \int_{\Omega} \mu \nabla w \cdot \nabla w v \, dx \quad \forall v \in V. \quad (11)$$

### 3.2 Compound scalar variational form

The alternative way of deriving the variational form is to introduce a test vector function  $\mathbf{v} \in \mathbf{V} = V \times V$  and take the inner product of  $\mathbf{v}$  and the residuals, integrated over the domain:

$$\int_{\Omega} (R_w, R_T) \cdot \mathbf{v} \, dx = 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

With  $\mathbf{v} = (v_0, v_1)$  we get

$$\int_{\Omega} (R_w v_0 + R_T v_1) \, dx = 0 \quad \forall \mathbf{v} \in \mathbf{V}.$$

Integrating the Laplace terms by parts results in

$$\int_{\Omega} (\mu \nabla w \cdot \nabla v_0 + \kappa \nabla T \cdot \nabla v_1) \, dx = \int_{\Omega} (\beta v_0 + \mu \nabla w \cdot \nabla w v_1) \, dx, \quad \forall \mathbf{v} \in \mathbf{V}. \quad (12)$$

Choosing  $v_0 = v$  and  $v_1 = 0$  gives the variational form (10), while  $v_0 = 0$  and  $v_1 = v$  gives (11).

With the inner product notation,  $(p, q) = \int_{\Omega} pq \, dx$ , we can alternatively write (10) and (11) as

$$\begin{aligned}(\mu \nabla w, \nabla v) &= (\beta, v) \quad \forall v \in V, \\ (\kappa \nabla T, \nabla v) &= (\mu \nabla w \cdot \nabla w, v) \quad \forall v \in V,\end{aligned}$$

or since  $\mu$  and  $\kappa$  are considered constant,

$$\mu(\nabla w, \nabla v) = (\beta, v) \quad \forall v \in V, \quad (13)$$

$$\kappa(\nabla T, \nabla v) = \mu(\nabla w \cdot \nabla w, v) \quad \forall v \in V. \quad (14)$$

### 3.3 Decoupled linear systems

The linear systems governing the coefficients  $c_j^{(w)}$  and  $c_j^{(T)}$ ,  $j = 0, \dots, N$ , are derived by inserting the expansions (7) in (10) and (11), and choosing  $v = \psi_i$  for  $i = 0, \dots, N$ . The result becomes

$$\sum_{j=0}^N A_{i,j}^{(w)} c_j^{(w)} = b_i^{(w)}, \quad i = 0, \dots, N, \quad (15)$$

$$\sum_{j=0}^N A_{i,j}^{(T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \dots, N, \quad (16)$$

$$A_{i,j}^{(w)} = \mu(\nabla\psi_j, \nabla\psi_i), \quad (17)$$

$$b_i^{(w)} = (\beta, \psi_i), \quad (18)$$

$$A_{i,j}^{(T)} = \kappa(\nabla\psi_j, \nabla\psi_i), \quad (19)$$

$$b_i^{(T)} = \mu\left(\left(\sum_j c_j^{(w)} \nabla\psi_j\right) \cdot \left(\sum_k c_k^{(w)} \nabla\psi_k\right), \psi_i\right). \quad (20)$$

It can also be instructive to write the linear systems using matrices and vectors. Define  $K$  as the matrix corresponding to the Laplace operator  $\nabla^2$ . That is,  $K_{i,j} = (\nabla\psi_j, \nabla\psi_i)$ . Let us introduce the vectors

$$b^{(w)} = (b_0^{(w)}, \dots, b_N^{(w)}),$$

$$b^{(T)} = (b_0^{(T)}, \dots, b_N^{(T)}),$$

$$c^{(w)} = (c_0^{(w)}, \dots, c_N^{(w)}),$$

$$c^{(T)} = (c_0^{(T)}, \dots, c_N^{(T)}).$$

The system (15)-(16) can now be expressed in matrix-vector form as

$$\mu K c^{(w)} = b^{(w)}, \quad (21)$$

$$\kappa K c^{(T)} = b^{(T)}. \quad (22)$$

We can solve the first system for  $c^{(w)}$ , and then the right-hand side  $b^{(T)}$  is known such that we can solve the second system for  $c^{(T)}$ .

### 3.4 Coupled linear systems

Despite the fact that  $w$  can be computed first, without knowing  $T$ , we shall now pretend that  $w$  and  $T$  enter a two-way coupling such that we need to derive the algebraic equations as *one system* for all the unknowns  $c_j^{(w)}$  and  $c_j^{(T)}$ ,  $j = 0, \dots, N$ . This system is nonlinear in  $c_j^{(w)}$  because of the  $\nabla w \cdot \nabla w$  product.

To remove this nonlinearity, imagine that we introduce an iteration method where we replace  $\nabla w \cdot \nabla w$  by  $\nabla w_- \cdot \nabla w$ ,  $w_-$  being the  $w$  computed in the previous iteration. Then the term  $\nabla w_- \cdot \nabla w$  is linear in  $w$  since  $w_-$  is known. The total linear system becomes

$$\sum_{j=0}^N A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^N A_{i,j}^{(w,T)} c_j^{(T)} = b_i^{(w)}, \quad i = 0, \dots, N, \quad (23)$$

$$\sum_{j=0}^N A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^N A_{i,j}^{(T,T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \dots, N, \quad (24)$$

$$A_{i,j}^{(w,w)} = \mu(\nabla \psi_j, \psi_i), \quad (25)$$

$$A_{i,j}^{(w,T)} = 0, \quad (26)$$

$$b_i^{(w)} = (\beta, \psi_i), \quad (27)$$

$$A_{i,j}^{(T,w)} = \mu((\nabla \psi w_-) \cdot \nabla \psi_j), \psi_i), \quad (28)$$

$$A_{i,j}^{(T,T)} = \kappa(\nabla \psi_j, \psi_i), \quad (29)$$

$$b_i^{(T)} = 0. \quad (30)$$

This system can alternatively be written in matrix-vector form as

$$\mu K c^{(w)} = b^{(w)}, \quad (31)$$

$$L c^{(w)} + \kappa K c^{(T)} = 0, \quad (32)$$

with  $L$  as the matrix from the  $\nabla w_- \cdot \nabla$  operator:  $L_{i,j} = A_{i,j}^{(w,T)}$ .

The matrix-vector equations are often conveniently written in block form:

$$\begin{pmatrix} \mu K & 0 \\ L & \kappa K \end{pmatrix} \begin{pmatrix} c^{(w)} \\ c^{(T)} \end{pmatrix} = \begin{pmatrix} b^{(w)} \\ 0 \end{pmatrix},$$

Note that in the general case where all unknowns enter all equations, we have to solve the compound system (23)-(24) since then we cannot utilize the special property that (15) does not involve  $T$  and can be solved first.

When the viscosity depends on the temperature, the  $\mu \nabla^2 w$  term must be replaced by  $\nabla \cdot (\mu(T) \nabla w)$ , and then  $T$  enters the equation for  $w$ . Now we have a two-way coupling since both equations contain  $w$  and  $T$  and therefore must be solved simultaneously. The equation  $\nabla \cdot (\mu(T) \nabla w) = -\beta$  is nonlinear, and if some iteration procedure is invoked, where we use a previously computed  $T_-$  in the viscosity ( $\mu(T_-)$ ), the coefficient is known, and the equation involves only one unknown,  $w$ . In that case we are back to the one-way coupled set of PDEs.

We may also formulate our PDE system as a vector equation. To this end, we introduce the vector of unknowns  $\mathbf{u} = (u^{(0)}, u^{(1)})$ , where  $u^{(0)} = w$  and  $u^{(1)} = T$ . We then have

$$\nabla^2 \mathbf{u} = \begin{pmatrix} -\mu^{-1} \beta \\ -\kappa^{-1} \mu \nabla u^{(0)} \cdot \nabla u^{(0)} \end{pmatrix}.$$

## 4 Different function spaces for the unknowns

It is easy to generalize the previous formulation to the case where  $w \in V^{(w)}$  and  $T \in V^{(T)}$ , where  $V^{(w)}$  and  $V^{(T)}$  can be different spaces with different numbers of degrees of freedom. For example, we may use quadratic basis functions for  $w$  and linear for  $T$ . Approximation of the unknowns by different finite element spaces is known as *mixed finite element methods*.

We write

$$\begin{aligned} V^{(w)} &= \text{span}\{\psi_0^{(w)}, \dots, \psi_{N_w}^{(w)}\}, \\ V^{(T)} &= \text{span}\{\psi_0^{(T)}, \dots, \psi_{N_T}^{(T)}\}. \end{aligned}$$

The next step is to multiply (5) by a test function  $v^{(w)} \in V^{(w)}$  and (6) by a  $v^{(T)} \in V^{(T)}$ , integrate by parts and arrive at

$$\int_{\Omega} \mu \nabla w \cdot \nabla v^{(w)} \, dx = \int_{\Omega} \beta v^{(w)} \, dx \quad \forall v^{(w)} \in V^{(w)}, \quad (33)$$

$$\int_{\Omega} \kappa \nabla T \cdot \nabla v^{(T)} \, dx = \int_{\Omega} \mu \nabla w \cdot \nabla v^{(T)} \, dx \quad \forall v^{(T)} \in V^{(T)}. \quad (34)$$

The compound scalar variational formulation applies a test vector function  $\mathbf{v} = (v^{(w)}, v^{(T)})$  and reads

$$\int_{\Omega} (\mu \nabla w \cdot \nabla v^{(w)} + \kappa \nabla T \cdot \nabla v^{(T)}) \, dx = \int_{\Omega} (\beta v^{(w)} + \mu \nabla w \cdot \nabla v^{(T)}) \, dx, \quad (35)$$

valid  $\forall \mathbf{v} \in \mathbf{V} = V^{(w)} \times V^{(T)}$ .

The associated linear system is similar to (15)-(16) or (23)-(24), except that we need to distinguish between  $\psi_i^{(w)}$  and  $\psi_i^{(T)}$ , and the range in the sums over  $j$  must match the number of degrees of freedom in the spaces  $V^{(w)}$  and  $V^{(T)}$ . The formulas become



$$\sum_{j=0}^{N_w} A_{i,j}^{(w)} c_j^{(w)} = b_i^{(w)}, \quad i = 0, \dots, N_w, \quad (36)$$

$$\sum_{j=0}^{N_T} A_{i,j}^{(T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \dots, N_T, \quad (37)$$

$$A_{i,j}^{(w)} = \mu(\nabla \psi_j^{(w)}, \psi_i^{(w)}), \quad (38)$$

$$b_i^{(w)} = (\beta, \psi_i^{(w)}), \quad (39)$$

$$A_{i,j}^{(T)} = \kappa(\nabla \psi_j^{(T)}, \psi_i^{(T)}), \quad (40)$$

$$b_i^{(T)} = \mu(\nabla w_-, \psi_i^{(T)}). \quad (41)$$

In the case we formulate one compound linear system involving both  $c_j^{(w)}$ ,  $j = 0, \dots, N_w$ , and  $c_j^{(T)}$ ,  $j = 0, \dots, N_T$ , (23)-(24) becomes

$$\sum_{j=0}^{N_w} A_{i,j}^{(w,w)} c_j^{(w)} + \sum_{j=0}^{N_T} A_{i,j}^{(w,T)} c_j^{(T)} = b_i^{(w)}, \quad i = 0, \dots, N_w, \quad (42)$$

$$\sum_{j=0}^{N_w} A_{i,j}^{(T,w)} c_j^{(w)} + \sum_{j=0}^{N_T} A_{i,j}^{(T,T)} c_j^{(T)} = b_i^{(T)}, \quad i = 0, \dots, N_T, \quad (43)$$

$$A_{i,j}^{(w,w)} = \mu(\nabla \psi_j^{(w)}, \psi_i^{(w)}), \quad (44)$$

$$A_{i,j}^{(w,T)} = 0, \quad (45)$$

$$b_i^{(w)} = (\beta, \psi_i^{(w)}), \quad (46)$$

$$A_{i,j}^{(T,w)} = \mu(\nabla w_- \cdot \nabla \psi_j^{(w)}, \psi_i^{(T)}), \quad (47)$$

$$A_{i,j}^{(T,T)} = \kappa(\nabla \psi_j^{(T)}, \psi_i^{(T)}), \quad (48)$$

$$b_i^{(T)} = 0. \quad (49)$$

The corresponding block form

$$\begin{pmatrix} \mu K^{(w)} & 0 \\ L & \kappa K^{(T)} \end{pmatrix} \begin{pmatrix} c^{(w)} \\ c^{(T)} \end{pmatrix} = \begin{pmatrix} b^{(w)} \\ 0 \end{pmatrix},$$

has square and rectangular block matrices:  $K^{(w)}$  is  $N_w \times N_w$ ,  $K^{(T)}$  is  $N_T \times N_T$ , while  $L$  is  $N_T \times N_w$ ,

## 5 Computations in 1D

We can reduce the system (5)-(6) to one space dimension, which corresponds to flow in a channel between two flat plates. Alternatively, one may consider

flow in a circular pipe, introduce cylindrical coordinates, and utilize the radial symmetry to reduce the equations to a one-dimensional problem in the radial coordinate. The former model becomes

$$\mu w_{xx} = -\beta, \tag{50}$$

$$\kappa T_{xx} = -\mu w_x^2, \tag{51}$$

while the model in the radial coordinate  $r$  reads

$$\mu \frac{1}{r} \frac{d}{dr} \left( r \frac{dw}{dr} \right) = -\beta, \tag{52}$$

$$\kappa \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) = -\mu \left( \frac{dw}{dr} \right)^2. \tag{53}$$

The domain for (50)-(51) is  $\Omega = [0, H]$ , with boundary conditions  $w(0) = w(H) = 0$  and  $T(0) = T(H) = T_0$ . For (52)-(53) the domain is  $[0, R]$  ( $R$  being the radius of the pipe) and the boundary conditions are  $du/dr = dT/dr = 0$  for  $r = 0$ ,  $u(R) = 0$ , and  $T(R) = T_0$ .

**Calculations to be continued...**

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