$$u'' + f(u') + s(u) = F(t)$$

Discretization

$$[D_tD_t u'' + f(D_{2t}u) + s(u) = F]^n$$

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + f\left(\frac{u^{n+1} - u^{n-1}}{2\Delta t}\right) + s(u^n) = F^n$$

u^{n+1}: unknown, nonlinear in u^{n+1} if f is nonlinear

Linear damping: $f(u') = bu'$, $b = \text{const} \geq 0$

$f(u') \approx b\dfrac{u^{n+1} - u^{n-1}}{2\Delta t}$, can solve wrt u^{n+1}

$\Rightarrow u^{n+1} = \ldots u^n, u^{n-1}$ explicit formula

Quadratic f: $f(u') = b|u'|u'$

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + b\left|\frac{u^{n+1} - u^{n-1}}{2\Delta t}\right|\frac{u^{n+1} - u^{n-1}}{2\Delta t} + s(u^n) = F^n$$

nonlinear in u^{n+1}.

Smart trick: approx $|u'|u'$ by a geometric mean

$$[|u'|u']^n \approx |u'|^{n-\frac{1}{2}}[u']^{n+1/2} + \mathcal{O}(\Delta t^2)$$

$$\approx \left|[D_t u]^{n-\frac{1}{2}}\right| [D_t u]^{n+\frac{1}{2}}$$

$$= \left|\frac{u^n - u^{n-1}}{\Delta t}\right| \cdot \frac{u^{n+1} - u^n}{\Delta t}$$ linear in u^{n+1} !!

$\Rightarrow u^{n+1} = \ldots\ldots u^n, u^{n-1} \ldots$ explicit scheme

Note: $s(u)$ is evaluated at t_n and is therefore known.

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$u^1 = ?$ Need to derive a special formula

scheme at $n=0, t=0$
initial cond $[D_{2t}u]^0 = V$ $\Bigg\} \Rightarrow u^1 = \ldots\ldots$

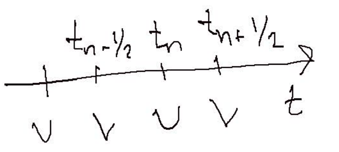
—

First-order formulation and staggered meshes:

First-order form:

$$\left\{\begin{array}{l} u' = v \\ v' = -f(v) - s(u) + F(t) \\ u(0) = I,\ v(0) = V. \end{array}\right\}$$

Staggered mesh: u^n, $v^{n+\frac{1}{2}}$



$t_{n-\frac{1}{2}}$ $\quad \dfrac{u^n - u^{n-1}}{\Delta t} = v^{n-\frac{1}{2}}$

t_n $\quad \dfrac{v^{n+\frac{1}{2}} - v^{n-\frac{1}{2}}}{\Delta t} = -f(v^n) - s(u^n) + F^n$

?

Linear f: $f(v) = -bv$, $v^n \approx \frac{1}{2}(v^{n-\frac{1}{2}} + v^{n+\frac{1}{2}})$ a la Crank-Nic.

arithmetic mean

Quadratic f: $f(v) = -b|v|v$, $[|v|v]^n \approx |v^{n-\frac{1}{2}}|\, v^{n+\frac{1}{2}}$

geometric mean

$$u^n = u^{n-1} + \Delta t\, v^{n-\frac{1}{2}}$$

$$\underbrace{v^{n+\frac{1}{2}} = v^{n-\frac{1}{2}} - \Delta t\, |v^{n-\frac{1}{2}}|\, v^{n+\frac{1}{2}}} - s(u^n) + F^n$$

$$v^{n+\frac{1}{2}} = \ldots\ldots$$