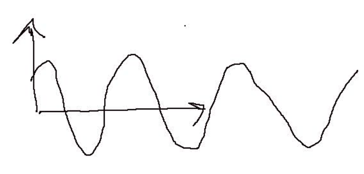
Last time: $u'' + \omega^2 u = 0$, $u(0) = I$, $u'(0) = V$



$u(t) = I \cos \omega t$, $V = 0$

Verification: Try a linear function for MMS.

$u_e = ct + d$, $u(0) = I \Rightarrow d = I$, $u'(0) = V \Rightarrow c = V$

Insert u_e in the ODE

$$0 + \omega^2 (Vt + d) \neq 0 \quad \text{add source term!}$$
$$= f$$

$\Rightarrow u'' + \omega^2 u = \omega^2 (Vt + d)$ exact analytical solution (conv. rate experiments)

Is the linear solution of the discrete equations?

$$u''(t_n) \approx [D_t D_t u]^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

$$\Rightarrow [D_t D_t u + \omega^2 u = f = \omega^2 (Vt + I)]^n \quad n \geq 1$$

Special formula for the first step:

scheme: $\dfrac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + \omega^2 u^0 = \omega^2 I \quad n = 0$

Initial cond: $u'(0) = V$ $[D_{2t} u]^0 = V$, $\dfrac{u^1 - u^{-1}}{2\Delta t} = V$

$$\Rightarrow u^{-1} = u^1 - 2\Delta t V$$

$$\Rightarrow u^1 = \underset{= I}{u^0} + \underbrace{\Delta t V}_{\text{new}} + \overbrace{\tfrac{1}{2}\Delta t^2 \omega^2 \underset{= I}{u^0} - \underbrace{\tfrac{1}{2}\Delta t \omega^2 I}_{\text{new}}}^{= 0}$$

$$u^{n+1} = 2u^n - u^{n-1} + \Delta t^2 \omega^2 u^n - \Delta t^2 \omega^2 (Vt_n + I), \quad n \geq 1$$

—

Check if $u^n = Vt_n + I = Vn\Delta t + I$ solves the two eq. for u^1 and u^n.

Can just insert and do the algebra, ...

We see that

$$u^{n+1} = 2u^n - u^{n-1} + 0$$

$$t_n + \Delta t = 2t_n - (t_n - \Delta t) \quad \text{fulfilled}$$

In general, check how differences work on polynomials:

$[D_t D_t t]^n = 0$, $[D_t D_t t^2]^n = 2t_n$ exact

$[D_t t]^n = 0$, $[D_{2t} t]^n = 0$

In such such cases, analytical derivative = finite difference ("scheme = ODE")

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Automation in sympy (or similar)

ODE: $L(u) = f$

Idea: guess u, set $f = L(u)$

Software: evaluate $L(u)$ (e.g. $L(u) = u'' - \omega^2 u$)

Discrete eq: $L_\Delta(u^n) - f^n = 0$

Software: evaluate $L_\Delta(u^n) - f^n$

Check: $L_\Delta(\underbrace{u(t_n)}_{\substack{\text{manufactured} \\ \text{chosen sol.}}}) - f^n = 0 \Rightarrow$ numerical sol = exact sol.

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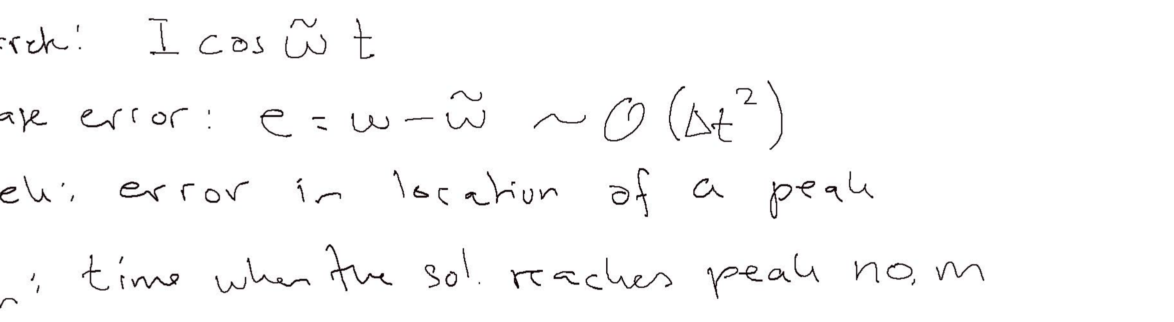
Exercise 1:



phase error

$\longrightarrow t$

Exact: $I \cos \omega t$

Discrete: $I \cos \tilde{\omega} t$

phase error: $e = \omega - \tilde{\omega} \sim \mathcal{O}(\Delta t^2)$

Seek: error in location of a peak

t_m: time when the sol. reaches peak no. m

location: $2\pi m$, t_m: $\omega t_m = 2\pi m \Rightarrow t_m = \dfrac{2\pi}{\omega} m$

Discrete: $\tilde{t}_m = \dfrac{2\pi}{\tilde{\omega}} m$

$$t_m - \tilde{t}_m = 2\pi m \left(\frac{1}{\omega} - \frac{1}{\tilde{\omega}} \right)$$

prop. to m

—

Energy consideration;

(Math approach)

$$u'' + \omega^2 u = 0, \quad u(0) = I, \quad u'(0) = V \quad \Big| \cdot u' dt \text{ and integrate}$$

$$\int_0^T u'' u' \, dt + \int_0^T \omega^2 u u' \, dt = 0$$

$$\frac{d}{dt} \frac{1}{2} (u')^2 \qquad \frac{d}{dt} \frac{1}{2} \omega^2 u^2$$

$$\Rightarrow \int_0^T \frac{d}{dt} \left(\frac{1}{2} (u')^2 + \frac{1}{2} \omega^2 u^2 \right) dt = 0$$

$$\Rightarrow \underbrace{\frac{1}{2} (u')^2 + \frac{1}{2} \omega^2 u^2}_{E(t)\text{ : total energy}} = \text{const}$$

$$\text{Const} = E(0) = \frac{1}{2} V^2 + \frac{1}{2} \omega^2 I^2$$

To verify: check that $E(t) = E(0)$.

Numerically $E(t)$ has an error $\mathcal{O}(\Delta t^2)$, but we can compute conv. rates

$$E^n = \frac{1}{2} \left([D_{2t} u]^n \right)^2 + \frac{1}{2} \omega^2 (u^n)^2$$

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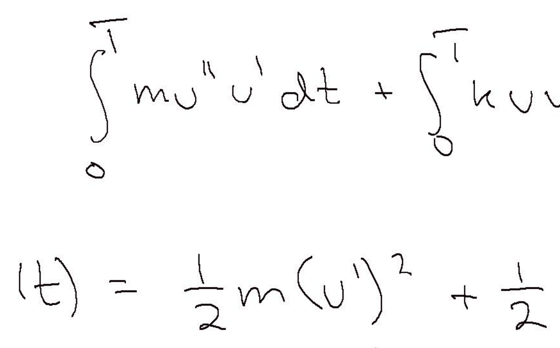
Physical derivation:



$$F = ma$$
$$\Downarrow$$
$$-ku = mu''$$

$$mu'' + ku = 0$$

Compute the work of each term (work $= \int_0^T (\,) \underbrace{u' dt}_{du}$)

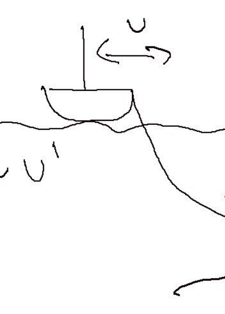
$$\int_0^T m u'' u' \, dt + \int_0^T k u u' \, dt = 0$$

$$E(t) = \underbrace{\frac{1}{2} m (u')^2}_{\substack{\text{kinetic} \\ \text{energy}}} + \underbrace{\frac{1}{2} k u^2}_{\substack{\text{potential} \\ \text{energy}}} = \text{const}$$

—

Generalized problem;

$$u'' + f(u') + s(u) = F(t), \quad u(0) = I, \quad u'(0) = V$$



$f \sim \frac{1}{2} C_D \varrho A |u'| u'$ air resitance

$S \sim \sin u$

F: environmental forces