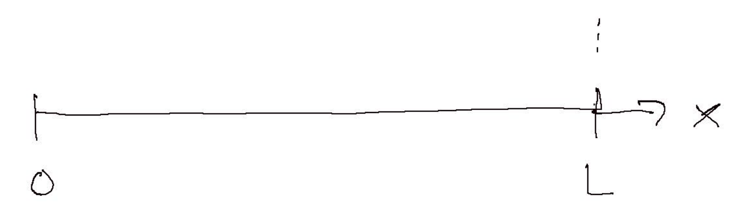
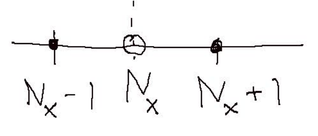


$$\frac{\partial u}{\partial x} = 0 \Rightarrow [D_{2x} u]^n_{N_x} = 0$$



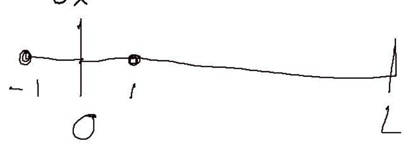
$$\frac{U^{N_x+1} - U^{N_x-1}}{2\Delta x} = 0$$

$$\Rightarrow U^{N_x+1} = U^{N_x-1}$$

We also use the scheme at $i=N_x$

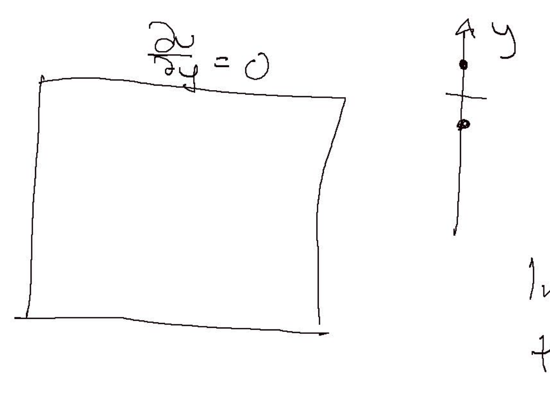
$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2} = c^2 \frac{\underbrace{U_{i+1}^n}_{= U_{i-1}^n} - 2U_i^n + U_{i-1}^n}{\Delta x^2} \qquad i = N_x$$

$\frac{\partial u}{\partial x} = 0$



$[D_{2x} u]^n_0 = 0 \Rightarrow U_{-1}^n = U_1^n$. In the scheme: $U_{i-1}^n = U_{i+1}^n$

2D: $\frac{\partial u}{\partial y} = 0$

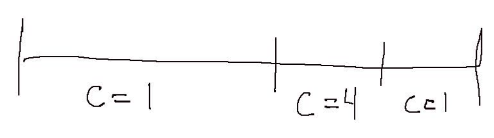


$$[D_{2y} u]^n_{N_y} = \frac{U^n_{N_y+1} - U^n_{N_y-1}}{2\Delta y} = 0$$

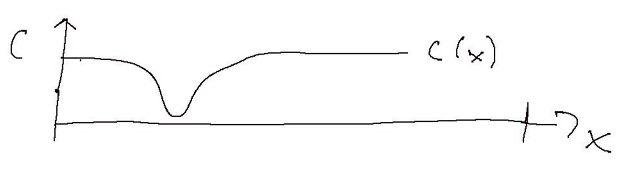
Insert $U^n_{N_y-1}$ for $U^n_{N_y+1}$ in the scheme

—

Variable wave velocity: $c = c(x)$



$$u_{tt} = \frac{\partial}{\partial x}\left(c^2(x) \frac{\partial u}{\partial x}\right)$$



$$u_{tt} = c^2(x) u_{xx} \quad \text{(alternative)}$$

Introduce $q(x) = c^2(x)$ for notational simplicity.

$$\frac{\partial}{\partial x}\left(q(x) \frac{\partial u}{\partial x}\right):$$

1. Discretize the outer $\frac{\partial}{\partial x}$: $\frac{\partial \varphi}{\partial x}$, $\varphi = q \frac{\partial u}{\partial x}$

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=x_i} \approx [D_x \varphi]^n_i = \frac{\varphi^n_{i+\frac{1}{2}} - \varphi^n_{i-\frac{1}{2}}}{\Delta x}$$

2. Discretize φ

$$\varphi^n_{i+1/2} = \left(q \frac{\partial u}{\partial x}\right)_{x=x_{i+1/2}} \approx q_{i+\frac{1}{2}} \frac{U^n_{i+1} - U^n_{i-1}}{\Delta x}$$

$$= [q D_x u]^n_{i+1/2}$$

$$\frac{\partial}{\partial x}\left(q \frac{\partial u}{\partial x}\right) = \frac{q_{i+\frac{1}{2}} \frac{U^n_{i+1} - U^n_{i-1}}{\Delta x} - q_{i-1/2} \frac{U^n_i - U^n_{i-1}}{\Delta x}}{\Delta x}$$

$$= [D_x (q D_x u)]^n_i$$

How to evaluate $q_{i+\frac{1}{2}}$?

- if $q(x)$ is known: $q_{i+\frac{1}{2}} = q(x_{i+\frac{1}{2}})$, $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$
- if q is known at the mesh points:

$$q_{i+\frac{1}{2}} \approx \frac{1}{2}(q_i + q_{i+1}) \quad \text{arithmetic average}$$

$$\frac{1}{q_{i+1/2}} \approx \frac{1}{2}\left(\frac{1}{q_i} + \frac{1}{q_{i+1}}\right) \quad \text{harmonic average (good for discontinuous } q(x)\text{)}$$

$$q_{i+1/2} \approx \sqrt{q_i q_{i+1}} \quad \text{geometric average}$$

Scheme with arithmetic average for q:

$$U_i^{n+1} = 2U_i^n - U_i^{n-1} + \frac{\Delta t^2}{\Delta x^2}\left(\frac{1}{2}(q_{i+1} + q_i)(U^n_{i+1} - U^n_i) - \frac{1}{2}(q_i + q_{i-1})(U^n_i - U^n_{i-1})\right)$$

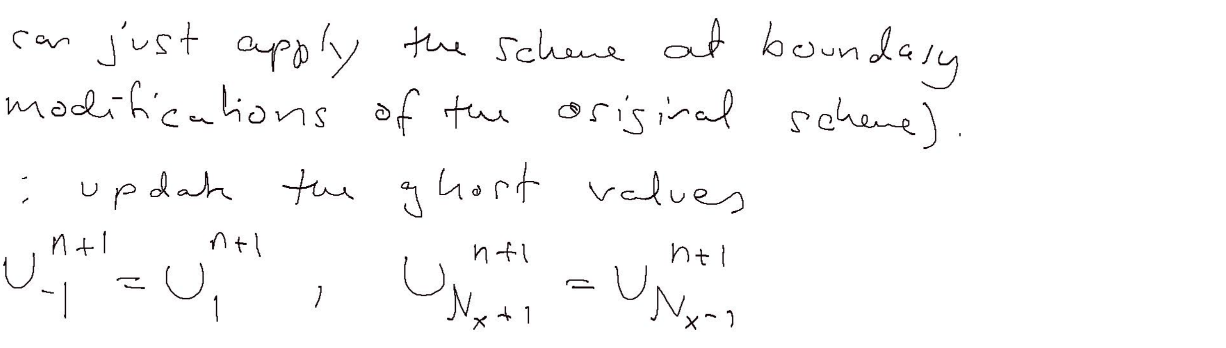
for $n \geq 1$ and all interior points ($i = 1, \ldots, N_x$)

$n=0$: special formula as usual ($U_t = 0 \Rightarrow U_i^{n-1} = U_i^{n+1}$)

Boundary conditions:
- $u=0$: insert $U_i^{n+1} = 0$ on the boundary
- $\frac{\partial u}{\partial x} = 0$: $[D_{2x} u] = 0 \Rightarrow U^n_{i+1} = U^n_{i-1}$, $i = N_x$
  $U^n_{i-1} = U^n_{i+1}$, $i = 0$

—

Ghost cells for implementing $\frac{\partial u}{\partial x} = 0$:



If $U_{-1}^n = U_1^n$ and $U^n_{N_x+1} = U^n_{N_x-1}$ then one can just apply the scheme at boundary (no modifications of the original scheme).

Then: update the ghost values

$$U_{-1}^{n+1} = U_1^{n+1}, \quad U^{n+1}_{N_x+1} = U^{n+1}_{N_x-1}$$

Now we are ready for the next time level

2D $\frac{\partial u}{\partial x} = 0$