Numerical errors

So far: looked at exact error expressions, e.g.

$u' = -au$ by FE $\quad u(0)=I$

$e^n = Ie^{-at_n} - I(1-a\Delta t)^n$ $\quad$ for a special

Alternative: compute convergence rates using a chosen exact solution (again: special cases)

Truncation error analysis: a general framework for deriving convergence rates (for general problems) by hand.

Abstract problem

$\mathcal{L}(u_e) = 0$ $\quad$ differential equation

$\mathcal{L}_\Delta(u) = 0$ $\quad$ discrete equations (scheme)

u_e: exact sol.
u: numerical sol.

Dream: derive an error estimate $e = u_e - u$
usually impossible

What we can: insert u_e i $\mathcal{L}_\Delta(u) = 0$

$$\mathcal{L}_\Delta(u_e) = \underset{\substack{\uparrow \\ \text{residual} \\ \text{or truncation error}}}{R} \neq 0$$

Given $\mathcal{L}_\Delta(u)$, we can Taylor series to derive an expression for R:

$$R = C\Delta t^{r} + D\Delta x^{p} + E\Delta y^{q}$$

conv. rate

involve derivatives of u_e

How accurate is the backward difference?

$$\frac{u^n - u^{n-1}}{\Delta t} = u'(t_n) + \underset{\uparrow e}{R^n}$$

Idea 1: assume Δt is small such that u^{n-1} can be represented by a Taylor series

Idea 2: expand series about $t = t_n$ (where we seek u')

$$u^{n-1} = u(t_{n-1}) = u(t_n) + u'(t_n)(-\Delta t) + \frac{1}{2}u''(t_n)(-\Delta t)^2 + \frac{1}{6}u'''(t_n)(-\Delta t)^3 + \mathcal{O}(\Delta t^4)$$

$$R^n = \frac{u^n - u^{n-1}}{\Delta t} - u'(t_n) \quad \text{(error in approximation)}$$

$$= \frac{u(t_n) - \left(u(t_n) + u'(t_n)(-\Delta t) + \frac{1}{2}u''(t_n)(-\Delta t)^2 + \mathcal{O}(\Delta t^3)\right)}{\Delta t} - u'(t_n)$$

$$= \cancel{u'(t_n)} - \frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2) - \cancel{u'(t_n)}$$

$$= -\frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2)$$

Truncation error $R^n \sim \Delta t$ (first order)

Can do the same for forward difference: $R^n = \frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2)$

See notes and slides for a collection of various formulas

ODE:

$$\frac{u^n - u^{n-1}}{\Delta t} = -au^n$$

We know:

$$\frac{u^n - u^{n-1}}{\Delta t} = u'(t_n) + R^n, \quad R^n = -\frac{1}{2}u''(t_n)\Delta t + \mathcal{O}(\Delta t^2)$$

Define the truncation error by inserting u_e:

$$\frac{u_e^n - u_e^{n-1}}{\Delta t} = -au_e^n + R^n$$ ← residual, error in equation = truncation error of the scheme

$$\Downarrow$$

$$u_e'(t_n) \underbrace{- \frac{1}{2}u_e''(t_n)\Delta t}_{R^n \text{ from the backward formula}} = -au_e^n + R^n$$

We know

$u_e'(t_n) = -au_e(t_n)$ $\quad (= -au_e^n)$ $\quad$ u_e solves the ODE!

$$\Rightarrow \quad R^n = -\frac{1}{2}u_e''(t_n)\Delta t + \mathcal{O}(\Delta t^2)$$ residual/trunc. error of the backward scheme

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Need: trunc. error of difference formulas.

forward $[D_t^+ u]^n = u'(t_n) + R^n, \quad R^n = \ldots\ldots$

centered $[D_t u]^n = u'(t_n) + R^n, \quad R^n = \ldots\ldots$

Can insert these in schemes to derive R^n.
See slides for automation by software and for many more examples!