$u'' + \omega^2 u = 0 \qquad u_e \sim I\cos\omega t$

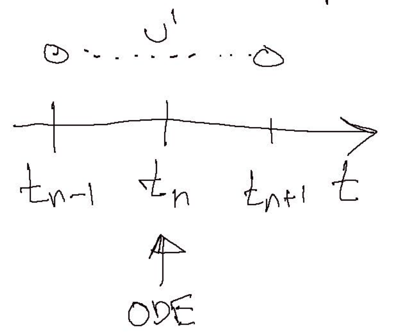
$u^n \sim I \cos\tilde{\omega} t$

$$\tilde{\omega} = \omega\left(1 + \frac{1}{24}\omega^2\Delta t^2\right) + \mathcal{O}(\Delta t^4)$$

$$u'' + \left(\omega - \frac{1}{24}\omega^2\Delta t^2\right)^2 u = 0 \qquad ?$$

make the freq. smaller because the scheme will increase it

Problem 4: Leapfrog scheme for $u' = -au + b$



$$u'(t_n) \approx \frac{u^{n+1} - u^{n-1}}{2\Delta t}$$

$$u'(t_n) = -a(t_n)u(t_n) + b(t_n)$$

Nonlinear version:

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = f(u^n) \qquad (u' = f(u))$$

unknown $\qquad$ known!

$$u^{n+1} = u^{n-1} + 2\Delta t f(u^n) \quad \text{explicit!!}$$

Much used in weather predictions.

Linear case

$$u^{n+1} = u^{n-1} - 2\Delta t\, a^n u^n + 2\Delta t\, b^n$$

$n=0$: problem - what is u^{-1}?

Use another scheme for u^1

Common choice: Forward Euler

$$u^1 = u^0 - \Delta t\, a^0 u^0 + \Delta t\, b^0.$$

u^2 = Leapfrog

$\vdots$

Result: $\dfrac{u^{n+1} - u^{n-1}}{2\Delta t} = [D_{2t}u]^n = u'(t_n) + \mathcal{O}(\Delta t^2)$

b) $u_e = ct + d$, $u_e(0) = I \Rightarrow d = I$, c free prm.

Inserted in the ODE

$$c = -a(ct + I) + b \Rightarrow b = c + a(ct + I)$$

c free const.

$a = a(t)$ free

Numerical scheme:

Forward Euler ($n=0$)

$$[D_t^+ u = -au + b]^n$$

$$[D_t^+(ct + I)]^n = c\underbrace{[D_t^+ t]^n}_{=1} + \underbrace{[D_t^+ I]}_{=0} \qquad \frac{I - I}{\Delta t} = 0$$

$$\frac{t + \Delta t - t}{\Delta t} = 1$$

$$\Rightarrow \underline{c} = -\underline{a^n(ct_n + I)} + \underline{c} + \underline{a^n(ct_n + I)} \qquad 0 = 0$$

Leapfrog scheme

$$[D_{2t}u = -au + b]^n$$

Critical issue: $[D_{2t}t]^n = 1$ ok.