Sist gang: ikke-lin. PDE $\longrightarrow$ lineær PDE

Nå: Diskretiser i tid og rom først, løs så ikke-lin. algebraiske likninger.

Modellproblem:

$$-(\alpha(u)u')' + au = f(u)$$

FDM:

$$[D_x \alpha D_x u + au = f]_i \qquad (\alpha(u) \text{ behandles som en } \alpha(x))$$

$$\Rightarrow \frac{1}{\Delta x^2}\left(\alpha_{i+1/2}(u_{i+1}-u_i) - \alpha_{i-1/2}(u_i-u_{i-1})\right) + au_i = f(u_i)$$

?

$$\alpha_{i+1/2} \approx \begin{cases} \alpha\left(\frac{1}{2}(u_i+u_{i+1})\right) = [\alpha(\bar{u}^x)]_i \\ \frac{1}{2}(\alpha(u_i)+\alpha(u_{i+1})) = [\overline{\alpha(u)}^x]_i \end{cases}$$

Picard: $\alpha(u_{-i})$ kjent verdi i α, $f(u_{-i})$ kjent verdi

Spesielle f: f.eks. $f(u) = u^4$, $f(u_i) \approx u_i \cdot \underbrace{u_{-i}^3}_{\text{kjent}}$

Newton:

$$F_i = \frac{1}{\Delta x^2}\left(\alpha_{i+1/2}(\ ) - \dots\right) - au_i - f(u_i) = 0$$

Jacobi-matrisen

$$J_{i,j} = \frac{\partial F_i}{\partial u_j}, \quad \text{kun bidrag for } j = i, i-1, i+1$$

$$J_{i,i-1} = \frac{\partial F_i}{\partial u_{i-1}} = \frac{1}{\Delta x^2}\left(-\frac{1}{2}(u_i-u_{i-1}) - \frac{1}{2}(u_i+u_{i-1})(-1)\right)$$

$$J_{i,i} = \dots \text{ mange ledd } \dots$$

FEM:

Variasjonsformulering av $-(\alpha u')' + au = f$

delv. int.

$$\int_0^L \alpha(u)u'v'\,dx + \int_0^L auv\,dx = \int_0^L f(u)v\,dx - \underbrace{\cancel{[\alpha(u)u'v]_0^L}}_{\text{fluks}}$$

Eksamen: $\to h(u-T_s)$

Variant 1: Prøver å regne ut de algebraiske likningene for hånd

Starter med $\int_0^L f(u)v\,dx$, $u = \sum c_j\psi_j$, $v = \psi_i$

$$\int \underbrace{f\left(\sum c_j\psi_j\right)}_{2}\psi_i\,dx$$

Selv med $f(u) = u^2$ blir det veldig komplisert

Program: numerisk integrasjon og vi har en approksimasjon u_- å putte inn i $f(u)$.

Group finite element method:

Ide: $u = \sum c_j\psi_j = \sum u_j\varphi_j$

$u(x_j)$: u i node j

$$f(u) \approx \sum f(u_j)\varphi_j(x)$$

$$\int_0^L f(u)\varphi_i\,dx \approx \int_0^L \sum \underbrace{f(u_j)}_{\text{tall}}\underbrace{\varphi_j \cdot \varphi_i}_{\text{avh. av } x}\,dx = \sum \underbrace{\left(\int \varphi_i\varphi_j\,dx\right)}_{\text{Massematrise}} f(u_j)$$

$$\int_0^L f(u)v\,dx \to M \cdot f$$

$$f = \begin{pmatrix} f(u_0) \\ \vdots \\ f(u_N) \end{pmatrix}$$

$$\int_0^L \alpha(u)u'v'\,dx \to \int_0^L \alpha\Big(\sum_k \underset{u_k}{c_k}\varphi_k\Big)\Big(\sum c_j\varphi_j'\Big)\varphi_i'\,dx \quad (c_i = u_i)$$

$$\underset{\text{group FEM}}{\approx} \sum_k\sum_j\left(\int_0^L \varphi_k\varphi_i'\varphi_j'\,dx\right)\alpha(u_k)u_j$$

Kan regne dette ut. Svaret blir

$$\int_0^L \alpha(u)u'v'\,dx \to h \cdot [D_x\overline{\alpha}^x D_x u]_i$$

Vi får m/group FEM:

$$\left[h\left(-D_x\overline{\alpha}^x D_x u + a\Big(u - \frac{h^2}{6}D_xD_xu\Big) - \Big(f(u) - \frac{h^2}{6}D_xD_xf(u)\Big)\right)\right]_i = 0$$

Massematrise, Massematrise, Samme

Trapesregel (kun i nodepkt): $= 0$, $= 0$

+ Picard el. Newton for å løse likningene

Variant 2: Jobber direkte med variasjonsformuleringen

$$\int_0^L (\alpha(u)u'v' + auv - f(u)v)\,dx = 0 \qquad \forall v \in V$$

$v = \psi_i$

$$F_i = \int_0^L (\alpha(u)u'\psi_i' + au\psi_i - f(u)\psi_i)\,dx = 0 \qquad i = 0, \dots, N$$

Picard it.: gamle verdier (u_-) i α og f:

$$F_i \approx \hat{F}_i = \int_0^L (\alpha(u_-)u'\psi_i' + au\psi_i - f(u_-)\psi_i)\,dx = 0$$

$\sum c_j\psi_j$

$\Rightarrow$ lineært likn.system "på vanlig måte"

Newton:

$$J(u_-)\,\delta u = -F(u_-)$$

F_i over, u_- insatt for u

$$J_{i,j} = \frac{\partial F_i}{\partial c_j}:$$

$$\frac{\partial}{\partial c_j}\int_0^L \alpha\Big(\sum c_k\psi_k\Big)\Big(\sum c_j\psi_j'\Big)\psi_i\,dx$$

u, u'

$$= \int_0^L \Big(\alpha'\Big(\sum c_k\psi_k\Big) \cdot \underbrace{\frac{\partial}{\partial c_j}\Big(\sum_k c_k\psi_k\Big)}_{=\psi_j} + \alpha\Big(\sum c_k\psi_k\Big)\underbrace{\frac{\partial}{\partial c_j}\Big(\sum_j c_j\psi_j'\Big)}_{=\psi_j'}\Big)\psi_i'\,dx$$

$$= \int_0^L (\alpha'(u)\psi_j + \alpha(u)\psi_j')\psi_i'\,dx$$

$$\frac{\partial}{\partial c_j}\int_0^L au\psi_i\,dx = \int_0^L a\underbrace{\frac{\partial}{\partial c_j}\Big(\overset{u}{\sum c_k\psi_k}\Big)}_{\psi_j}\psi_i\,dx = \int_0^L a\psi_i\psi_j\,dx$$

$$\frac{\partial}{\partial c_j}\int_0^L f(u)\psi_i\,dx = \int_0^L \frac{\partial}{\partial c_j}f\Big(\sum_k c_k\psi_k\Big)\psi_i\,dx$$

$$= \int_0^L f'(u)\underbrace{\frac{\partial}{\partial c_j}\Big(\sum c_k\psi_k\Big)}_{\psi_j}\psi_i\,dx$$

$$= \int_0^L f'(u)\psi_i\psi_j\,dx$$

Har nå hele $J_{i,j}$.

Generalisert til 2D/3D:

$$-\nabla \cdot \alpha(u)\nabla u + au = f(u)$$

$$F_i = \int (\alpha(u)\nabla u \cdot \nabla\psi_i + au\psi_i - f(u)\psi_i)\,dx = 0$$

$u = \sum c_k\psi_k$

$$J_{i,j} = \frac{\partial F_i}{\partial c_j} = \int \Big((\alpha'(u)\psi_j\nabla u + \alpha(u)\nabla\psi_j)\cdot\nabla\psi_i + a\psi_j\psi_i - f'(u)\psi_j\psi_i\Big)\,dx$$

Husk: kjent u_- skal benyttes for u i $J_{i,j}$ og F_i i Newtons metode.