Ikke-lineære ODE & PDE

Teknikker:

- Eksplisitt diskretisering
- Geometrisk middel i tid $\quad \left((u^2)^{n+\frac{1}{2}} \approx \underset{\text{kjent}}{u^n} u^{n+1}\right)$
- Picard iterasjon
- Newton's metode

PDE:

$$u_t = \nabla\cdot(\alpha(u)\nabla u) + f(u)$$

Forward Euler:

$$\frac{u^{n+1}-u^n}{\Delta t} = \underbrace{\nabla\cdot\alpha(u^n)\nabla u^n + f(u^n)}_{\text{kjent!}} \qquad \Delta t \le \frac{\Delta x^{2}}{\max \alpha} \quad \text{Problem}$$

Backward Euler:

$$\frac{u^n-u^{n-1}}{\Delta t} = \nabla\cdot\alpha(u^n)\nabla u^n + f(u^n)$$

Picard på PDE-nivå

$u : u^n$, u_- : approksimasjon til u^n fra forrige iterasjon.
$(u_- = u^{n-1}$ i starten av iterasjonen)

$u_1 : u^{n-1}$

$$\frac{u-u_1}{\Delta t} = \nabla\cdot\alpha(u)\nabla u + f(u)$$

Picard: lineariser ikke-lin. ledd vha. u_- (a la $u^2 \approx u_-\cdot u$)

$$\frac{u-u_1}{\Delta t} = \nabla\cdot\alpha(u_-)\nabla u + f(u_-) \qquad (*)$$

Men: spesiell struktur på $f(u)$, f.eks. $u(1-u)$,

$f(u) \approx u(1-u_-)$

$\sin(u) \approx u - \frac{1}{6}u_-^2 u$ (hvis u er liten)

Merk: $(*)$ er en lineær PDE i rommet

Variasjonsformulering:

$$\int_\Omega \frac{u-u_1}{\Delta t} v\,dx = -\int_\Omega \alpha(u_-)\nabla u\cdot\nabla v\,dx + \int_\Omega f(u_-)v\,dx$$
$$+ \underbrace{\int_{\partial\Omega} \alpha(u_-)\frac{\partial u}{\partial n} v\,ds}_{\text{avh. randbet.}}$$

Newton på PDE nivå:

$$u = u_- + \underbrace{\delta u}_{\text{korreksjon}}$$

$$\frac{u_-+\delta u-u_1}{\Delta t} = \nabla\cdot\alpha(u_-+\delta u)\nabla(u_-+\delta u) + f(u_-+\delta u)$$

Taylor-linearisering

$$\alpha(u_-+\delta u) = \alpha(u_-) + \alpha'(u_-)\delta u + \cancel{\mathcal{O}(\delta u^2)}$$

tilsv. for $f(u_-+\delta u) \approx f(u_-) + f'(u_-)\delta u$

Innsatt:

$$\frac{u_-+\delta u-u_1}{\Delta t} = \nabla\cdot\left(\alpha(u_-)+\alpha'(u_-)\delta u\right)\nabla(u_-+\delta u)$$
$$+ f(u_-) + f'(u_-)\delta u$$

Pga. liten δu dropper vi alle ikke-lin. ledd i δu

$$\frac{u_-+\delta u-u_1}{\Delta t} = \nabla\cdot\alpha(u_-)\nabla u_- + \nabla\cdot\alpha(u_-)\nabla\delta u$$
$$+ \nabla\cdot\alpha'(u_-)\delta u\nabla u_- + f(u_-) + f'(u_-)\delta u$$

δu: ukjent

$$\frac{\delta u}{\Delta t} = \nabla\cdot\alpha(u_-)\nabla\delta u + \nabla\cdot\alpha'(u_-)\delta u\nabla u_- + f'(u_-)\delta u$$
$$+ \text{ ledd med } u_- \text{ (kjente)}$$

FDM el FEM i rommet.

Løs mhp δu, $\quad u_- := u_- + \delta u \quad \begin{pmatrix} u = u_- + \delta u \\ u_- = u \end{pmatrix}$

1D problem: $\quad -(\alpha(u)u')' + au = f(u)$

Differansemetoden:

$$-\left[D_x \overline{\alpha(u)}^{x} D_x u + au = f(u)\right]_i \qquad \text{optional}$$

Uten midling av α

$$D_x\alpha D_x u = \frac{1}{h^2}\left(\alpha_{i+\frac{1}{2}}(u_{i+1}-u_i) - \alpha_{i-\frac{1}{2}}(u_i-u_{i-1})\right)$$

$$\alpha_{i+\frac{1}{2}} = \alpha(u_{i+\frac{1}{2}}) \approx \begin{cases} \alpha\left(\frac{1}{2}(u_i+u_{i+1})\right) \\ \frac{1}{2}\left(\alpha(u_i)+\alpha(u_{i+1})\right) \end{cases}$$

To middel

$\left[D_x\overline{\alpha(u)}^{x}D_x\right]_i$ el. $\left[D_x\alpha(\bar{u}^{x})D_x\right]_i$

(FY: $(\alpha u')' = \alpha' u' + \alpha u''$ Gjør aldri dette! kan få problemer med $\alpha' u'$)

Ikke-lin. algebraiske likninger:

$$F_i = \frac{-1}{\Delta x^2}\Big(\frac{1}{2}(\alpha(u_i)+\alpha(u_{i+1}))(u_{i+1}-u_i) -$$
$$\frac{1}{2}(\alpha(u_{i-1})+\alpha(u_i))(u_i-u_{i-1})\Big) + au_i - f(u_i) = 0$$

Picard: gamle verdier i α og $f \Rightarrow$ lineært problem

Newton: Jacobianen

$$J_{i,j} = \frac{\partial F_i}{\partial u_j} \qquad J_{i,i-1} = \frac{\partial F_i}{\partial u_{i-1}} = -\frac{1}{\Delta x^2}\Big(\frac{1}{2}\alpha'(u_{i-1})(u_i-u_{i-1}) -$$
$$\frac{1}{2}\alpha(u_{i-1}) + \alpha(u_i)\Big)$$

$J_{i,i}$ = mange ledd

$J_{i,i+1} = \dfrac{\partial F_i}{\partial u_{i+1}}$ ca som $\dfrac{\partial F_i}{\partial u_{i-1}}$

$J_{i,j} = 0$ hvis $j > i+1$, $j < i-1 \Rightarrow J_{i,j}$ er tridiagonal

I Newton systemet

$$\sum_j J_{i,j}\,\delta u_j = -F_i\,, \quad i = 0, 1, \ldots$$

kjente verdier for u_{i-1}, u_i, u_{i+1} i J og F!

Alternativ: Picard og Newton på PDE-nivå og deretter diskretisere

$$-(\alpha(u)u')' + au = f(u)$$

Picard: $\quad -(\alpha(u_-)u')' + au = f(u_-)$

Diskretiser med FDM (samme som over, men lineært fordi vi har satt inn u_-)

Newton: $\quad u = u_- + \delta u$

$\alpha(u_-+\delta u) \approx \alpha(u_-) + \alpha'(u_-)\delta u$
tilsv. for f
dropper ledd med δu^2

Diskretiser med FDM — får de samme lineære algebraiske likningene som dukket opp i Newton-systemet

$$\sum_j J_{i,j}\,\delta u_j = -F_i$$

Diskretisering med FEM.

$$-(\alpha(u)u')' + au = f(u)$$

Variasjonsformulering:

$$\int_0^L \alpha(u)u'v'\,dx - \underbrace{\left[\alpha(u)u'v\right]_0^L}_{\text{avh. av randbet.}} + \int_0^L auv\,dx = \int_0^L f(u)v\,dx$$

Picard: u_- i $\alpha(u)$ og $f(u)$

$$\int_0^L \alpha(u_-)u'v'\,dx - [\;] + \int_0^L auv\,dx = \int_0^L f(u_-)v\,dx$$

Lineært variasjonsproblem $\quad a(u,v) = L(v)$

$$a(u,v) = \int_0^L \alpha(u_-)u'v'\,dx + \int_0^L auv\,dx$$
$$L(v) = \int_0^L f(u_-)v\,dx + [\;]$$

Newton: Trenger $F_i = 0$ og $J_{i,j} = \dfrac{\partial F_i}{\partial c_j}$, $\; u = \sum_j c_j\psi_j$

F_i : $v = \psi_i$

$$F_i = \int_0^L \alpha(u)u'\psi_i'\,dx + \int_0^L au\psi_i\,dx - \int_0^L f(u)\psi_i\,dx = 0$$

$\sum_j c_j\psi_j$

$$J_{i,j} = \frac{\partial F_i}{\partial c_j}\,, \quad \text{husk } u = \sum_j c_j\psi_j\,, \quad \boxed{\frac{\partial u}{\partial c_j} = \psi_j}$$

$$J_{i,j} = \int_0^L \Big(\alpha'(u)\underbrace{\frac{\partial u}{\partial c_j}}_{\psi_j} u'\psi_i' + \alpha(u)\underbrace{\frac{\partial u'}{\partial c_j}}_{\psi_j'}\psi_i'\Big)dx + \int_0^L a\frac{\partial u}{\partial c_j}\psi_i\,dx$$
$$+ \int_0^1 f'(u)\underbrace{\frac{\partial u}{\partial c_j}}_{\psi_j}\psi_i\,dx$$

Formelen for F_i brukes for element vektor.
Formelen for $J_{i,j}$ — " — element matrise.
Assemblerer og får

$$J\,\delta u = -F$$

Sett inn u_- for u

$$u = u_- + \delta u \quad (\text{el. } u = u_- + \omega\,\delta u)$$

Alternativ: Newton på PDE-nivå, så FEM diskr.

$u = u_- + \delta u$, Taylor/linearisering i δu

Likn. med δu (som ukjent) og u_- (kjent)

$$-(\alpha(u_-)\delta u')' + a\,\delta u = -(\alpha(u_-)u_-')' + f(u_-) + \ldots\ldots$$

Variasjonsformulering: $\quad \delta u = \sum_j c_j\psi_j$

$$\int_0^L \alpha(u_-)\,\delta u'\,v'\,dx + \int_0^L a\,\delta u\,v\,dx = \int_0^L \alpha(u_-)u_-'v'\,dx + \ldots$$

$F_i = 0$: $v = \psi_i$, $\delta u = \sum c_j\psi_j$, $u_- = \sum_j (c_j)_-\psi_j$

$$J_{i,j} = \frac{\partial F_i}{\partial c_j} = \int_0^L \alpha(u_-)\underbrace{\frac{\partial\,\delta u'}{\partial c_j}}_{\psi_j'} v'\,dx + \ldots$$

$$= \int_0^L \alpha(u_-)\psi_j'\psi_i'\,dx + \ldots$$

Fin oppgave:

PDE + variasjonsformulering + Newton $= \ldots$
$\quad ?$
PDE + Newton + variasjonsform. + Jacobian $= \ldots$