$u' = -au$, $u(0) = I$ $\quad u_e(t) = Ie^{-at}$

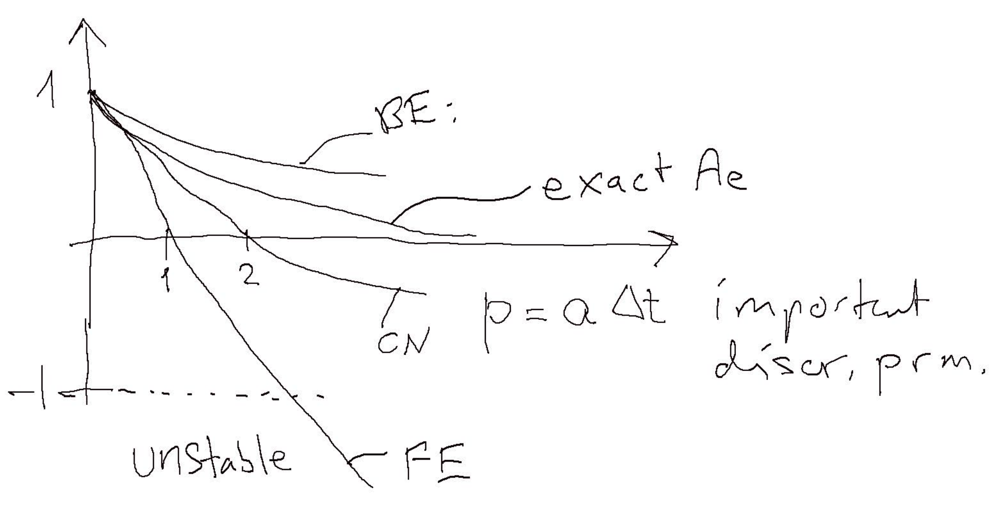
Accuracy: many approaches

Amplification factor

$$U^{n+1} = AU^n, \quad A = \frac{1-(1-\theta)a\Delta t}{1+\theta a \Delta t}$$

$$u_e(t_{n+1}) = A_e u_e(t_n) = \underbrace{Ie^{-a\Delta t}}_{A_e} \cdot u_e(t_n)$$

Compare A vs. A_e



Compare A vs. A_e analytically

$A_e - A = ?$ Taylor series! Sympy!

$$A_e - A \sim \begin{cases} \mathcal{O}(\Delta t^2) & \text{FE, BE} \\ \mathcal{O}(\Delta t^3) & \text{CN} \end{cases}$$

This is a local error measure.

Alternative: $1 - \frac{A}{A_e}$. Same results

Better: The global (or true) error at a point

$$E = u_e(t_n) - U^n = Ie^{-at_n} - IA^n$$

Method: Taylor series expansion in sympy

$$E \sim \begin{cases} \Delta t, & \text{FE, BE} \\ \Delta t^2, & \text{CN} \end{cases} \quad \text{at one point}$$

Alternative measure of the global error: a norm

$$E = \left(\int_0^T (u_e - U^n)^2 dt\right)^{1/2}$$

must be extended with values between the mesh points

In practice: numerical integration

$$E = \left(\Delta t \sum_{n=0}^{N_t} (e^n)^2\right)^{1/2}$$

$$e^n = u_e(t_n) - U^n \quad \text{error mesh function}$$

Compute analytically (see the notes)

Conclusion:

$$E \sim \begin{cases} \Delta t, & \text{FE, BE} \\ \Delta t^2, & \text{CN} \end{cases}$$

The order (1st vs 2nd) stays the same for the global error at a point and the integrated global error.

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Truncation error:

How good is the discrete eq.? (easy to find an answer)

$$[D_t^+ u = -au]^n \quad \text{(FE)}$$

What happens if we insert the exact solution in this eq?

$$[D_t^+ u_e + au_e = R]^n$$

error in the discr. eq. = truncation error

Taylor expansion of u_e around t_n

some calc.

$$\Rightarrow \quad R^n = \underbrace{u_e'(t_n) + au_e(t_n)}_{=0 \text{ because it solves the ODE}} + \frac{1}{2}u_e''(t_n)\Delta t + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow \quad R^n \sim \frac{1}{2}u_e''\Delta t \qquad \text{error} \sim \Delta t$$

$$\text{Truncation error} \sim \begin{cases} \Delta t, & \text{FE, BE} \\ \Delta t^2, & \text{CN} \end{cases}$$

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Convergence: global/true error $\to 0$ as $\Delta t \to 0$ ("right solution")

Consistency: truncation error $\to 0$ as $\Delta t \to 0$ ("right equation")

Stability: correct qualitative behavior of U^n

The Lax theorem for linear differential equations

consistency + stability $\Leftrightarrow$ convergence

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