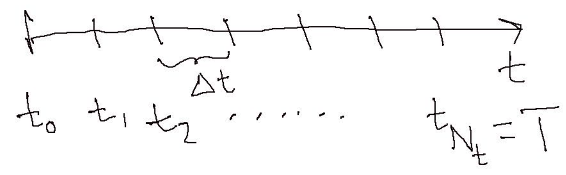
Finite difference methods

4 steps!

Model problem: $u'(t) = -au(t)$, $u(0) = I$

Step 1: Discretizing the domain $t \in (0, T]$



Step 2: Sample the ODE at the mesh points

$$u'(t_n) = -au(t_n), \quad n = 1, 2, \dots N_t$$

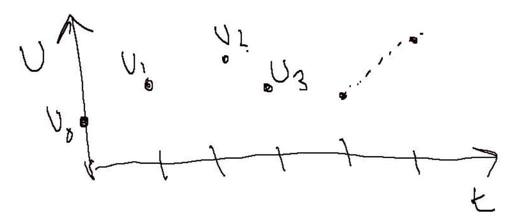
Step 3: Replace derivatives by finite differences

E.g.

$$u'(t_n) \approx \frac{u(t_{n+1}) - u(t_n)}{\Delta t} \qquad \text{Forward difference}$$

Notation: $u^n \equiv u(t_n)$ numerical solution

Mesh function:



Inserting the finite diff. approx:

$$\frac{u^{n+1} - u^n}{\Delta t} = -au^n \qquad \text{difference scheme (equation)}$$

Step 4: Derive the final algorithm

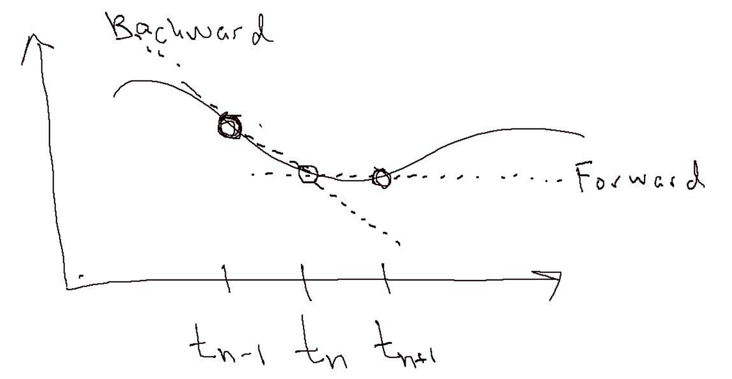
Idea: u^n is known, u^{n+1} is unknown

Solve wrt. u^{n+1}:

$$u^{n+1} = u^n - a\Delta t\, u^n \qquad \text{The Forward Euler scheme}$$

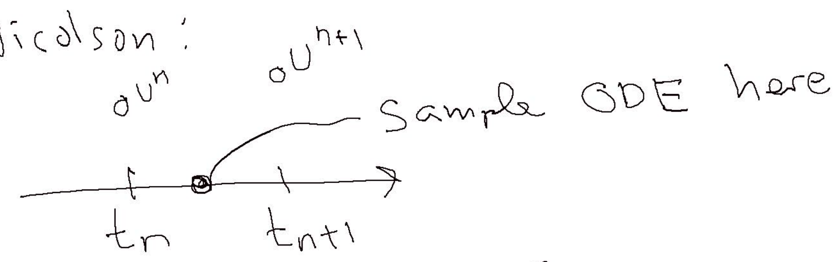
$$u^0 = I$$

Other schemes



$$u'(t_n) \approx \frac{u^{n+1} - u^{n-1}}{2\Delta t} \qquad \text{leapfrog diff.}$$

Crank-Nicolson:



$$u'(t_{n+1/2}) \approx \frac{u^{n+1} - u^n}{\Delta t}$$

$$u'(t_{n+1/2}) = -au(t_{n+1/2}) \approx -a\frac{1}{2}(u^n + u^{n+1})$$

$$\Rightarrow \frac{u^{n+1} - u^n}{\Delta t} = -a\frac{1}{2}(u^n + u^{n+1})$$

Solve wrt u^{n+1}:

$$u^{n+1} = \frac{1 - \frac{1}{2}\Delta t a}{1 + \frac{1}{2}\Delta t a} u^n, \quad u^0 = I$$

Nice combination of Crank-Nicolson, Forward and Backward Euler schemes:

$$u^{n+1} = \frac{1 - (1-\theta)\Delta t a}{1 + \theta \Delta t a} u^n \qquad \theta\text{-rule}$$

$\theta = 0$: Forward Euler (FE)

$\theta = \frac{1}{2}$: Crank-Nicolson (CN)

$\theta = 1$: Backward Euler (BE)